Examples handout

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0.1 Setup.

Fix $n \geq 2$, $G = \mathrm{GL}_n$, T the usual torus, $X^*(\hat{T}) = \mathbf{Z}^n$ as usual, $\alpha_i \in X^*(\hat{T})$ the usual roots with $1 \leq i \leq n-1$.

We know that generalized Steinberg representations of $G(\mathbf{Q}_p)$ are in canonical bijection with subsets $I \subset \{1, 2, \dots, n-1\}$. Explicitly, given $I = \{i_1 < \cdots < i_k\}$, let P_I be the standard parabolic with Levi $\mathrm{GL}_{i_1} \times \mathrm{GL}_{i_2 - i_1} \times \cdots \times \mathrm{GL}_{i_k - i_{k-1}} \times \mathrm{GL}_{n-i_k}$, and let π_I be the unique irreducible quotient of $\mathcal{C}(P_I(\mathbf{Q}_p)\backslash G(\mathbf{Q}_p))$. Under this bijection, $I = \emptyset$ gives the trivial representation and $I = \{1, 2, \dots, n-1\}$ gives the Steinberg representation.

More generally, pick any $0 \leq d < n$ (or really any d), and let $b_{d/n} = b_d$ be the basic isocrystal of slope d/n. Write $n' = n/(d,n)$ and $d' = d/(d,n)$, where we declare $(0,n) = n$. Then $G_{b_d} = \mathrm{GL}_{(d,n)}(D_{\frac{d'}{n'}})$ is an inner form of G, and generalized Steinberg representations of G_{b_d} are in canonical bijection with subsets $I \subset \{1, 2, \dots, (d,n) - 1\}$, by the same recipe as above. When $I = \{1, 2, \dots, (d,n)-1\}$, $\rho_I = \mathrm{St}_{G_{b_d}}$ is the Steinberg representation.

Let I^c denote the complement of I in $\{0 < j < (d,n)\}$, and let I^t denote the transpose ($j \in I$ iff $(d,n) - j \in I^t$). One can show that $\rho_I^\vee = \rho_{I^t}$ and $\mathrm{Zel}(\rho_I) = \rho_{I^c}$.

0.2 Examples

By an old calculation of Drinfeld, we have

$$R\Gamma_c(\Omega^1, \mathbf{Q}_\ell) = \mathrm{St}[-1] \oplus \mathbf{1}[-2].$$

This was generalized by Schneider and Stuhler: for any $n \geq 2$ we have

$$R\Gamma_c(\Omega^{n-1}, \mathbf{Q}_\ell)[n-1] = \mathrm{St} \oplus \pi_{\{2,3,\dots,n-1\}}[-1] \oplus \pi_{\{3,4,\dots,n-1\}}[-2] \oplus \cdots \oplus \pi_{\{n-1\}}[2-n] \oplus \mathbf{1}[1-n].$$

In modern terminology, Ω^{n-1} is the basic Newton stratum in the flag variety $\mathbf{P}^{n-1} = \mathrm{Gr}(1,n)$. What about the basic Newton strata in other flag varieties?

Take $\mathrm{Gr}(2,4)$. Then the (normalized) cohomology of $\mathrm{Gr}(2,4)^{\mathrm{bas}}$ is

$$\mathrm{St}[1] \oplus \pi_{\{1,3\}} \oplus \pi_{\{1\}}[-1] \oplus \pi_{\{3\}}[-1] \oplus \mathbf{1}[-2] \oplus \mathbf{1}[-4].$$

Same for $\mathrm{Gr}(2,5)$. Then the cohomology is

$$\mathrm{St}^2 \oplus \pi_{\{2,3,4\}}[-1] \oplus \pi_{\{1,2,4\}}[-1] \oplus \pi_{\{1,2\}}[-2] \oplus \pi_{\{2,4\}}[-2]$$
$$\oplus \pi_{\{2\}}[-3] \oplus \pi_{\{4\}}[-3] \oplus \mathbf{1}[-4] \oplus \mathbf{1}[-6].$$

Same for $\mathrm{Gr}(2,6)$. Then the cohomology is

$$\begin{aligned} \mathrm{St}[1]^2 \oplus \pi_{\{1,2,4,5\}} \oplus \pi_{\{2,3,4,5\}} \oplus \pi_{\{1,2,5\}}[-1] \oplus \pi_{\{2,4,5\}}[-1] \\ \oplus\pi_{\{1,2\}}[-2] \oplus \pi_{\{2,5\}}[-2] \oplus \pi_{\{4,5\}}[-2] \oplus \pi_{\{2\}}[-3] \oplus \pi_{\{5\}}[-3] \\ \oplus\mathbf{1}[-4] \oplus \pi_{\{5\}}[-5] \oplus \mathbf{1}[-6] \oplus \mathbf{1}[-8]. \end{aligned}$$

Same for $\mathrm{Gr}(3,6)$. The cohomology is

$$\begin{aligned} \mathrm{St}[2]^2 \oplus \pi_{\{1,2,3,5\}}[1] \oplus \pi_{\{1,3,4,5\}}[1] \\ \oplus\pi_{\{1,2,3\}} \oplus \pi_{\{1,3,5\}} \oplus \pi_{\{3,4,5\}} \oplus \pi_{\{1,3\}}[-1] \oplus \pi_{\{1,5\}}[-1] \oplus \pi_{\{3,5\}}[-1] \\ \oplus\pi_{\{1\}}[-2] \oplus \pi_{\{1\}}[-4] \oplus \pi_{\{3\}}[-2] \oplus \pi_{\{5\}}[-2] \oplus \pi_{\{5\}}[-4] \\ \oplus\mathbf{1}[-3] \oplus \mathbf{1}[-5]^2 \oplus \mathbf{1}[-7] \oplus \mathbf{1}[-9]. \end{aligned}$$

In these calculations, we're really computing $i_1^* T_{\wedge^d \mathrm{std}^\vee} i_{b_{d/n}!}\mathbf{1}$. In fact, we can compute any composition $i_{b_{d/n}}^* T_{V_\mu} i_{b_{e/n}!}\rho$ where ρ is a generalized Steinberg of $G_{b_{e/n}}$ and $\mu = (w_1, \ldots, w_n)$ is any dominant weight. Note that this composition vanishes unless $d = e + \sum_{1\le i\le n} w_i$. Here are some more examples:

For any GL_n,

$$i_{b_{1/n}}^* T_{\mathrm{std}} i_{1!}\mathrm{St} = \mathbf{1}^n$$

(this was previously known).

On GL_6,

$$\begin{aligned} i_{b_{3/6}}^* T_{\wedge^3 \mathrm{std}} i_{1!}\mathrm{St} = \mathrm{St}^{14} \oplus \rho_{\{1\}}[-1]^2 \oplus \rho_{\{2\}}[-1]^2 \\ \oplus \mathbf{1}[-2] \oplus \mathbf{1}[-4] \end{aligned}$$

and

$$\begin{aligned} i_1^* T_{\wedge^3 \mathrm{std}^\vee} i_{b_{3/6}!}\mathrm{St} = \mathrm{St}^5 \oplus \pi_{\{1,2,3,4\}}[-1]^2 \oplus \pi_{\{2,3,4,5\}}[-1]^2 \oplus \pi_{\{1,2,4,5\}}[-1] \\ \oplus \pi_{\{1,2,4\}}[-2] \oplus \pi_{\{2,3,4\}}[-2] \oplus \pi_{\{2,4,5\}}[-2] \\ \oplus \pi_{\{1,2\}}[-3] \oplus \pi_{\{2,4\}}[-3] \oplus \pi_{\{4,5\}}[-3] \\ \oplus \pi_{\{2\}}[-4] \oplus \pi_{\{4\}}[-4] \oplus \mathbf{1}[-5] \oplus \mathbf{1}[-7]. \end{aligned}$$

On GL_8,

$$\begin{aligned} i_{b_{4/8}}^* T_{\mathrm{std}} i_{b_{3/8}!}\mathbf{1} = \mathrm{St}^2 \oplus \rho_{\{1,2\}}[-1]^2 \oplus \rho_{\{1\}}[-2] \\ \oplus \rho_{\{1\}}[-4] \oplus \mathbf{1}[-5] \oplus \mathbf{1}[-7]. \end{aligned}$$

On GL_9,

$$\begin{aligned} i_{b_{3/9}}^* T_{\mathrm{std}} i_{b_{2/9}!}\mathbf{1} = \mathrm{St}^3 \oplus \rho_{\{1\}}[-1] \oplus \rho_{\{1\}}[-3]^2 \\ \oplus \mathbf{1}[-4] \oplus \mathbf{1}[-6] \oplus \mathbf{1}[-8]. \end{aligned}$$