Problem list

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This is a list of open problems and questions. For a problem to appear on here, it has to meet several loose criteria:

1. It is not famous ("Prove the Hodge conjecture") and not grandiose to the point of intimidation ("Formulate and prove a precise p-adic categorical local Langlands correspondence").
2. It's interesting to me.
3. It can plausibly be attacked with current technology.

However, some of these problems are surely very hard. I will do my best to keep this document updated if (when!) problems are resolved.

p-adic geometry

1.1. (H.-Mann) Let $\mathcal{O}(\lambda)$ denote the usual vector bundle on the Fargues-Fontaine curve. For fixed integers $d < n$, let $W_{n,d}$ be the moduli space parametrizing quotient bundles $\mathcal{O}(1/n) \twoheadrightarrow \mathcal{E}$ such that $\mathcal{E}$ is isomorphic to $\mathcal{O}(1/d)$ at all geometric points. Prove that $W_{n,d}$ is a perfectoid space.

Remarks. This is true when $d = 1$ by the main result of Johansson-Ludwig (with a proof by global methods!), and the arguments in the appendix of that paper handle the case $d = n - 1$.

1.2. (H.-Scholze) Prove that all local Shimura varieties are Stein spaces.

1.3. Let $\tilde{X} \to X$ be a proetale G-torsor for some p-adic Lie group G, where X is a smooth rigid space in characteristic zero and $\tilde{X}$ is a perfectoid space. Prove that if $\tilde{X}$ is affinoid perfectoid, then X is an affinoid rigid space.

Remarks. This would settle the previous problem in all EL and PEL cases.

1.4. (H.-Kedlaya) Let A be a uniform Tate ring over $\mathbf{Q}_p$, and assume that all completed residue fields of A at all points $x \in \mathrm{Spa}A$ are perfectoid fields. Prove that the global sections of $\hat{\mathcal{O}}$ on the v-site of $\mathrm{Spa}A$ are perfectoid.

Remarks. In combination with the work of Tongmu He, this would imply that all local and global Shimura varieties at infinite level are perfectoid spaces.

1.5. Find genuinely new ways of constructing non-algebraizable smooth proper rigid spaces, besides the usual constructions involving Hopf surfaces, abeloids, and nonalgebraizable deformations of K3 surfaces. For example, does there exist a rigid space X admitting a formal model whose special fiber is a threefold of general type, and which admits no nonconstant meromorphic functions?

1.6. Let X/C be a connected smooth proper rigid space, and let $\mathbf{V}$ be a proetale $\mathbf{Q}_p$-local system on X. By the work of Anschutz-Le Bras-Mann, the cohomology groups $H^i(X, \mathbf{V})$ are (the C-points of) Banach-Colmez spaces, so they have well-defined C-dimensions and $\mathbf{Q}_p$-dimensions.

Prove that

$$\sum_i (-1)^i \dim_C H^i(X, \mathbf{V}) = 0.$$

Remarks. This is trivially true if $\mathbf{V}$ admits a $\mathbf{Z}_p$-lattice, and I checked it by hand in many nontrivial cases, e.g. if X is a smooth projective variety and $\mathbf{V}$ is pulled back from the rank n Lubin-Tate local system on $\mathbf{P}^{n-1}_C$ via some embedding $X \hookrightarrow \mathbf{P}^{n-1}_C$. For the complementary question on $\mathbf{Q}_p$-dimensions, I have a proof of the formula

$$\begin{aligned}\sum_i (-1)^i \dim_{\mathbf{Q}_p} H^i(X, \mathbf{V}) &= \mathrm{rank}(\mathbf{V}) \sum_i (-1)^i \dim H^i(X, \mathbf{Q}_p) \\ &= \mathrm{rank}(\mathbf{V}) \sum_{i,j} (-1)^{i+j} h^{i,j}(X)\end{aligned}$$

where the second equality follows from the primitive comparison theorem and the Hodge-Tate spectral sequence.

1.7. Let X/C and $\mathbf{V}$ be as in the previous problem. By the work of Anschutz-Le Bras-Mann, there is a canonical perfect complex $\mathcal{E}(\mathbf{V})$ on $\mathrm{FF}_{C^\flat}$ such that $R\Gamma(X, \mathbf{V}) = R\Gamma(\mathrm{FF}_{C^\flat}, \mathcal{E}(\mathbf{V}))$. Since $\mathrm{FF}_{C^\flat}$ is a regular scheme of dimension one, there is a splitting $\mathcal{E}(\mathbf{V}) \simeq \oplus \mathcal{E}^i(\mathbf{V})[-i]$ where $\mathcal{E}^i = H^i(\mathcal{E})$ is a coherent sheaf.

Is it true that for $i \geq \dim X$, all Harder-Narasimhan slopes of $\mathcal{E}^i(\mathbf{V})$ are nonnegative?

1.8. Let S be a perfectoid space, with relative Fargues-Fontaine curve X_S. Following Anschutz-Le Bras, we can define a (relatively) flat coherent sheaf $\mathcal{E}/X_S$ by the requirement that locally on S, $\mathcal{E}$ is isomorphic to the cokernel of a fiberwise-injective map of vector bundles $\mathcal{E}_1 \to \mathcal{E}_2$. If S is a point, any such $\mathcal{E}$ sits in a canonical short exact sequence $0 \to \mathcal{E}^{\mathrm{tors}} \to \mathcal{E} \to \mathcal{E}^{\mathrm{vb}} \to 0$ where the first term is a torsion sheaf and the third term is a vector bundle. We can then define the HN polygon of $\mathcal{E}$ as the concave-down piecewise-linear polygonal segment beginning with a vertical segment joining $(0,0)$ to $(0, \mathrm{length}(\mathcal{E}^{\mathrm{tors}}))$, and then continuing with the usual segments of the HN polygon of $\mathcal{E}^{\mathrm{vb}}$. In other words, the torsion part contributes a vertical segment slope $+\infty$ with length equal to the torsion length.

Prove that for any S and any relatively flat $\mathcal{E}/X_S$, the function sending $s \in |S|$ to $\mathrm{HN}(\mathcal{E}_s)$ is upper semicontinuous.

Remarks. For a simple but instructive example, it is easy to write down a family of injective maps $\mathcal{O} \to \mathcal{O}(1)^2$ over the base $\mathrm{Spd} C \left\langle T^{1/p^\infty} \right\rangle$ whose (relatively flat) cokernel away from $T = 0$ is $\mathcal{O}(2)$, but whose cokernel at $T = 0$ is $i_{\infty *} C \oplus \mathcal{O}(1)$.

When $\mathcal{E}$ is a vector bundle, this semicontinuity is a fundamental result of Kedlaya-Liu (with a nice reproof in Fargues-Scholze). In Hamann-H.-Scholze, we proved that the torsion length of $\mathcal{E}_s$ can only jump on a closed subset.

1.9. Let $C/\mathbf{Q}_p$ be a complete algebraically closed field, and consider Banach-Colmez spaces over C. By Colmez's fundamental results, any BC space $\mathbb{V}$ has a C-dimension and a $\mathbf{Q}_p$-dimension. By work of Plut, the slope function $\mu(\mathbb{V}) = -\frac{\dim_{\mathbf{Q}_p} \mathbb{V}}{\dim_C \mathbb{V}} \in \mathbf{Q} \cup \{-\infty\}$ defines a Harder-Narasimhan formalism on BC spaces. In particular, we can define the HN polygon of any BC space, with the proviso that the BC space $\underline{\mathbf{Q}_p}^m$ contributes a vertical segment of slope $-\infty$ and length m. Note that if $\mathcal{O}(\lambda)$ is the usual stable vector bundle on X_C, then the BC spaces $H^0(\mathcal{O}(\lambda))$ (for positive λ) and $H^1(\mathcal{O}(\lambda))$

(for negative λ) are stable of slope $-\frac{1}{\lambda}$. If $\mathcal{F}$ is a torsion coherent sheaf on X_C, then $H^0(\mathcal{F})$ is semistable of slope zero.

It is less clear how to define BC spaces in families. Here is a proposal. Let S be a perfectoid space, with relative Fargues-Fontaine curve X_S. Following ideas of Le Bras and Anschutz-Le Bras, define a BC space over S as an object $\mathcal{E} \in \mathrm{Perf}(X_S)$ which, locally in the analytic topology on S, is quasi-isomorphic to a two-term complex $[\mathcal{E}^{-1} \to \mathcal{E}^0]$ in degrees -1 and 0, where $\mathcal{E}^{-1}$ resp. $\mathcal{E}^0$ is a vector bundle with only negative resp. nonnegative slopes at all geometric points of S. These form a full subcategory $\mathrm{BC}(S)$ of $\mathrm{Perf}(X_S)$, and the association $S \mapsto \mathrm{BC}(S)$ is a v-stack. When $S = \mathrm{Spa}C$ is a geometric point, $H^*(X_C, \mathcal{E})$ is concentrated in degree zero and is (the C-points of) a usual BC space, and Le Bras proved that the evident functor $\mathcal{E} \mapsto H^0(X_C, \mathcal{E})$ is an equivalence of categories towards BC spaces in Colmez's original sense. For general S, the results of Anschutz-Le Bras imply that derived pushforward along $\tau : X_{S,v} \to S_v$ sends any BC space to a v-sheaf concentrated in degree zero and defines a full embedding of $\mathrm{BC}(S)$ into the category of v-sheaves of $\mathbf{Q}_p$-vector spaces on S. In particular, this equips $\mathrm{BC}(S)$ with the structure of an exact category (though for general S it is certainly not abelian).

Aside: I believe one can show that $\mathcal{E} \in \mathrm{Perf}(X_S)$ is a BC space in the above sense *if and only if* $R\tau_*\mathcal{E} \in D(S_v, \mathbf{Q}_p)$ is concentrated in degree zero.

Over general bases, the C-dimension and $\mathbf{Q}_p$-dimension make sense as locally constant functions on the base: if $\mathbb{V}$ is presented by a complex $[\mathcal{E}^{-1} \to \mathcal{E}^0]$ as above, then $\dim_C \mathbb{V} = \deg\mathcal{E}^0 - \deg\mathcal{E}^{-1}$ and $\dim_{\mathbf{Q}_p} \mathbb{V} = \mathrm{rank}\mathcal{E}^0 - \mathrm{rank}\mathcal{E}^{-1}$ are well-defined invariants of $\mathbb{V}$. However, families of BC spaces can exhibit some surprising behavior. For example, one can exhibit a family degenerating from $H^0(\mathcal{F})$, with $\mathcal{F}$ a length two torsion sheaf, to $H^0(\mathcal{O}(1)) \oplus H^1(\mathcal{O}(-1))$.

Problems:

i. If $\mathbb{V}$ is a BC space over S, prove that the function sending $s \in |S|$ to the HN polygon of $\mathbb{V}_s$ is upper semicontinuous.

ii. Prove that the v-stack $\mathcal{BC}$ sending S to $\mathrm{BC}(S)$ is an Artin v-stack, cohomologically smooth of pure dimension zero, with connected components given by the substacks $\mathcal{BC}^{d,h}$ parametrizing BC's with C-dimension d and $\mathbf{Q}_p$-dimension h for any fixed integers $(d,h) \in \mathbf{Z}_{\geq 0} \times \mathbf{Z}$.

iii. Prove that $\mathcal{BC}^{(0,h)} \simeq */\mathrm{GL}_h(\mathbf{Q}_p)$ is an isolated point for all $h \geq 0$, and that these are the only isolated points.

iv. Prove that the closure of any HN stratum in $\mathcal{BC}$ is a union of HN strata.

v. Prove that any HN stratum in $\mathcal{BC}^{(d,h)}$, with corresponding polygon P, is ℓ-cohomologically smooth of ℓ-dimension $-2A$, where A is the area enclosed between the polygon P and the straight line joining $(0,0)$ to (d,h).

Smooth representation theory

In these problems, G is a connected reductive group over a nonarchimedean local field E with residue field $\mathbf{F}_q$.

2.1. Let G be a quasisplit reductive group over an equal characteristic local field E. Let $W_\psi = \mathrm{c-ind}_{U(E)}^{G(E)}\psi$ be the Whittaker representation attached to a Whittaker datum (B, ψ), and let $W_{\psi,\mathfrak{s}}$ be the summand of W_ψ corresponding to a Bernstein component $\mathfrak{s}$. Prove that

i. W_ψ is projective,

ii. $W_{\psi,\mathfrak{s}}$ is finitely generated,

iii. $\mathbf{D}_{\mathrm{coh}} W_{\psi,\mathfrak{s}} \simeq W_{\psi^{-1},\mathfrak{s}^\vee}$

Remarks. These results are known for E of mixed characteristic (i. by Chan-Savin, ii. by Bushnell-Henniart, iii. by H.).

2.2. Let π be a smooth irreducible representation, $M(\pi)$ its standard module. Let σ be a field automorphism of $\mathbf{C}$ preserving $\sqrt{q}$. Prove that $M(\pi^\sigma) \simeq M(\pi)^\sigma$.

Remarks. This is known for GL_n (exercise via the Zelevinsky classification) and for quasisplit classical groups with E of characteristic zero (Koshikawa-Shin).

2.3. Assume E has characteristic zero. Let π be a tempered irreducible representation, with Harish-Chandra character Θ_π. Consider the function $\Theta_\pi^{\mathrm{st}} : G(E)_{\mathrm{ell}} \to \mathbf{C}$ sending an elliptic element g to $\frac{1}{|[[g]]|} \sum_{g' \in [[g]]} \Theta_\pi(g)$, where $[[g]]$ denotes the finite set of conjugacy classes in the stable conjugacy class of g. Prove that if Θ_π^{st} is not identically zero, then π is a discrete series representation.

Remarks. The converse direction is true (a terse account is given in Proposition 2.4 here). This is known for quasisplit classical groups, but only as a full consequence of the endoscopic classification. An a priori proof would allow for some soft and uniform construction of stable discrete series L-packets.

2.4. Let G be a quasisplit group with a fixed Whittaker datum. Assume some form of the refined local Langlands correspondence. If π corresponds to an enhanced parameter (ϕ, ρ), find a clean and explicit method/algorithm to compute the enhanced parameter of the Aubert-Zelevinsky dual $\hat{\pi}$.

Remarks. For GL_n (where the enhancement plays no role) this is still a nontrivial problem, but there is a satisfying answer by Moeglin-Waldspurger (in terms of Zelevinsky's classification via multisegments). There is also a later paper of Knight-Zelevinsky which explicates the resulting "multisegment duality" more clearly.

In general, the CFMMX paper suggests that in Vogan's formulation of LLC, the AZ involution roughly corresponds to taking the Fourier transform of the associated simple perverse sheaf. However, this doesn't solve the problem: computing the Fourier transform is not easy.

Categorical local Langlands and related things

3.1. Understand categorical local Langlands fully around the $(q^{-1}, 1, 1, q)$-parameter for GL_4, and around the infamous S_3-parameter for G_2. This could mean, among other things, fully computing the stalks of eigensheaves, explicitly writing down the coherent sheaves attached to irreducible representations and/or their standard modules at basic points, and fully understanding the automorphic sheaves corresponding to the structure sheaf and other irreducible vector bundles on the relevant Vogan stack.

3.2. Suppose b is basic and π is a discrete series irreducible representation of $G_b(E)$. Is it true that CLLC matches $i_{b!}\pi$ with a coherent sheaf on Par_G concentrated in degree zero?

3.3. Find an intrinsic characterization of the t-structure on $\mathrm{IndCoh}(\mathrm{Par}_G)$ which matches the perverse t-structure on $D(\mathrm{Bun}_G)$. Is there a description in terms of the enhanced fibers $i_z^{!\mathrm{enh}}$? (See section 6.1 of Arinkin-Gaitsgory for this notion.)

3.4. Prove that the Fargues-Scholze map $\pi \mapsto \varphi_\pi$ has finite fibers without appealing to known results on the local Langlands correspondence.

3.5. Prove the exact analogue of Theorem 6.5.2 in H.-Kaletha-Weinstein for local fields of equal characteristic p.

3.6. Can Atobe's results on Jacquet modules be related to CLLC?

3.7. Formulate a precise conjectural matching between period sheaves and L-sheaves under CLLC, in the sense of Ben Zvi-Sakellaridis-Venkatesh, and relate it to known conjectures and results on distinction.

Remarks. Some preliminary computations for the Hecke period on GL_2 indicate that getting the shears and twists correct will be somewhat annoying.

3.8. Find some direct relationship between the Fargues-Scholze geometry and endoscopy.

3.9. Assuming some form of the Tate conjecture over $\overline{\mathbf{F}_q}$, there is a realization functor from Voevodsky motives over $\overline{\mathbf{F}_q}$ towards some derived category $\mathcal{D}_{\mathbf{R}}$ of $\mathbf{R}$-vector spaces with some additional structure. (Precisely: we can take $\mathcal{D}_{\mathbf{R}} = \mathrm{Ind}D^b(\mathrm{Kt}_{\mathbf{R}})$, where $\mathrm{Kt}_{\mathbf{R}}$ is the category of finite-dimensional $\mathbf{Z}$-graded $\mathbf{C}$-vector spaces $V = \oplus_{i\in\mathbf{Z}} V_i$ equipped with a graded conjugate-linear isomorphism $\alpha : V \overset{\sim}{\to} V$ such that $\alpha^2 = (-1)^i$ on V_i. See e.g. Section 9 of Scholze's 2018 ICM lecture, and Iakovenko's thesis.) This induces a "real realization functor"

$$D_{\mathrm{mot}}(\mathrm{Bun}_G) \to D_{\mathrm{mot}}(\mathrm{Bun}_G) \otimes_{D_{\mathrm{mot}}(\overline{\mathbf{F}_q})} \mathcal{D}_{\mathbf{R}}.$$

What is the meaning of this realization? What does it correspond to on the spectral side? Note that the right-hand side is semiorthogonally glued from pieces which are just smooth $G_b(E)$-representations in $\mathcal{D}_{\mathbf{R}}$, for b varying over $B(G)$ as usual, so this is quite close to the dream of "doing Fargues-Scholze with real/complex coefficients".

Miscellany

4.1. Formulate and prove a precise form of geometric Satake with almost $\mathcal{O}^+/p$ coefficients on the B_{dR} affine Grassmannian.

Remarks. Even for GL_2 this would be very interesting.

4.2. Formulate a precise categorical local Langlands conjecture with mod-p or p-torsion coefficients. Ideally, for $\mathrm{GL}_2/\mathbf{Q}_p$ there should be some provable direct relationship with Colmez's work.

4.3. Let $\mathscr{X}$ be an eigenvariety as in e.g. my thesis. Is it true that any reduced irreducible component of $\mathscr{X}$ is Cohen-Macaualay?

Remarks. "Patched eigenvariety" techniques probably have something to say about this in many cases.

4.4. Prove the local weight-monodromy conjecture for nearby cycles of the constant sheaf on smooth rigid spaces without any auxiliary assumptions. In other words, remove the extra conditions on the formal model in H.-Zavyalov Theorem 1.4.2.

4.5. Is there a formula for Deligne-Langlands ϵ-factors in mixed characteristic which involves the words "Fourier transform" and "Banach-Colmez space", and is roughly analogous to the work of Laumon in equal characteristic? Ideally, such a formula would imply all the known formal properties of ϵ-factors.