Excursion operators and the stable Bernstein center

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Abstract

We prove that the Fargues-Scholze construction of elements in the Bernstein center via excursion operators always yields stable distributions. We also prove a strong quantitative compatibility of the Fargues-Scholze construction with transfer across extended pure inner forms. The proofs combine the character formulas from [HKW22], the commutation of Hecke operators with excursion operators, an averaging trick due to Fu [Fu24], and Arthur's theory of elliptic tempered virtual characters. The arguments work uniformly for all connected reductive groups over p-adic local fields.

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1 Introduction

1.1 Main results

Fix a finite extension $E/\mathbf{Q}_p$, and let G/E be a connected reductive group. Let $\mathfrak{Z}(G)$ be the Bernstein center of G, regarded as the convolution algebra of essentially compact invariant distributions on $G(E)$. This acts by convolution on the space of functions $C_c(G(E))$, and on all smooth $G(E)$-modules via the identification of $\mathfrak{Z}(G)$ with the center of the category. If π is a smooth irreducible $G(E)$-representation, we write z_π for the scalar with $z \cdot \pi = z_\pi \pi$.

Following the usual terminology, we say a distribution $z \in \mathfrak{Z}(G)$ is *stable* if it vanishes on unstable functions f, i.e. on functions with vanishing stable orbital integrals. These form a submodule $\mathfrak{Z}^{\mathrm{st}}(G) \subset \mathfrak{Z}(G)$. We say a distribution z is *very stable* if $z * f$ is unstable for all unstable f. Although this condition (first singled out by Scholze-Shin [SS13]) is a priori more restrictive than stability, in practice it is much easier to verify. It is easy to see that very stable distributions are stable, and that they form a commutative subalgebra $\mathfrak{Z}^{\mathrm{vst}}(G) \subset \mathfrak{Z}(G)$ such that $\mathfrak{Z}^{\mathrm{st}}(G)$ is naturally a $\mathfrak{Z}^{\mathrm{vst}}(G)$-module.

These stability conditions are expected to play a key role in local harmonic analysis and the local Langlands correspondence. Indeed, it is a by-now-standard conjecture that stable and very stable elements of $\mathfrak{Z}(G)$ coincide, and that z is (very) stable if and only if for all tempered L-packets $\Pi_\phi(G)$, $z_{\pi_1} = z_{\pi_2}$ for all $\pi_1, \pi_2 \in \Pi_\phi(G)$. For groups with a sufficiently well-understood local Langlands correspondence, this conjecture was recently proved by Varma in a beautiful paper [Var24, Theorem 4.4.2]. However, the second part of this conjecture certainly doesn't make any sense without prior knowledge of the local Langlands correspondence, and it seems extremely hard to construct (very) stable central elements from scratch (see [BKV15, BKV16] for some interesting results in this direction). The importance of constructing elements of the stable center for global purposes has previously been emphasized by Haines [Hai14], who also highlighted the expected connection with algebraic functions on the variety of semisimple L-parameters.

In this paper we show that the Fargues-Scholze machinery is perfectly suited to the construction of very stable central distributions. More precisely, let $\mathfrak{Z}^{\mathrm{spec}}(G)$ be the ring of global functions on the variety of semisimple L-parameters for G. In their amazing paper [FS24], Fargues-Scholze constructed a canonical ring map $\Psi_G : \mathfrak{Z}^{\mathrm{spec}}(G) \to \mathfrak{Z}(G)$ satisfying a long list of compatibilities, using V. Lafforgue's formalism of excursion operators [Laf18] adapted to the Fargues-Fontaine curve.[1] To streamline the discussion, let us write $\mathfrak{Z}^{\mathrm{FS}}(G)$ for the image of Ψ_G. Our first main result is the following theorem, essentially confirming a conjecture of Haines [Hai14] and Scholze-Shin [SS13, Conjecture 6.3].

Theorem 1.1. *The map $\Psi_G : \mathfrak{Z}^{\mathrm{spec}}(G) \to \mathfrak{Z}(G)$ factors over the subalgebra of very stable central distributions. Equivalently, there is an inclusion $\mathfrak{Z}^{\mathrm{FS}}(G) \subseteq \mathfrak{Z}^{\mathrm{vst}}(G)$.*

We emphasize that G is completely arbitrary. While we expect the inclusion $\mathfrak{Z}^{\mathrm{FS}}(G) \subseteq \mathfrak{Z}^{\mathrm{vst}}(G)$ is an equality for all groups, this seems far out of reach.

This theorem has several corollaries. First, recall that a virtual character $\Theta = \sum_{1 \leq i \leq j} a_i \Theta_{\pi_i}$ is *atomically stable* if Θ is stable, with all coefficients $a_i \neq 0$, and no smaller linear combination $\sum_{i \in I \subsetneq [1,j]} b_i \Theta_{\pi_i}$ is stable.

Corollary 1.2. *If $\Theta = \sum a_i \Theta_{\pi_i}$ is an atomically stable virtual character, the Fargues-Scholze parameter φ_{π_i} is independent of i.*

Now suppose G splits over a tame extension and $p \nmid |W_G|$. Then for any regular supercuspidal parameter $\phi : W_E \to {}^L G$, Kaletha [Kal19] explicitly constructed a supercuspidal L-packet $\Pi_\phi(G)$. By work of Fintzen-Kaletha-Spice [FKS23], the linear combination $S\Theta_\phi = \sum_{\pi \in \Pi_\phi(G)} \Theta_\pi$ is atomically stable. The previous corollary then immediately gives the following result.

Corollary 1.3. *For varying $\pi \in \Pi_\phi(G)$, the Fargues-Scholze parameter φ_π depends only on ϕ.*

Of course, we expect that $\varphi_\pi = \phi$, but this seems to be a very difficult problem.

More generally, our main result immediately shows that for any group G for which the *existence of tempered L-packets for G* is known in the precise sense of [Var24, Hypothesis 2.5.1], the Fargues-Scholze parameter is constant on any such packet. By the results of [Art13, Mok15], this condition is satisfied for all quasisplit classical groups.

[1] As written, [FS24] in fact defines an analogous map $\mathfrak{Z}^{\mathrm{spec}}(G, \overline{\mathbf{Q}_\ell}) \to \mathfrak{Z}(G, \overline{\mathbf{Q}_\ell})$ for any fixed prime $\ell \neq p$ and any fixed algebraic closure $\overline{\mathbf{Q}_\ell}$, where the source is the ring of functions on the variety of semisimple L-parameters into ${}^L G(\overline{\mathbf{Q}_\ell})$, and the target is the center of the category of smooth $\overline{\mathbf{Q}_\ell}$-representations of $G(E)$. In this paper, we simply transport this map across a fixed choice of isomorphism $\iota : \overline{\mathbf{Q}_\ell} \xrightarrow{\sim} \mathbf{C}$, for some fixed $\ell \neq p$. However, by recent work of Scholze [Sch25], the resulting map is completely canonical and independent of the choices involved.

We can also say something about how the image of the map Ψ_G changes as G varies across inner forms. To explain this, note that $\mathfrak{Z}^{\mathrm{spec}}(G)$ depends only on the inner isomorphism class of G. In particular, if G^* is quasisplit and $b \in B(G^*)$ is a basic element with associated extended pure inner form $G := G^*_b$, Theorem 1.1 gives a pair of maps

$$\begin{tikzcd} & \mathfrak{Z}^{\mathrm{vst}}(G^*) \\ \mathfrak{Z}^{\mathrm{spec}}(G^*) \arrow[ur, "\Psi_{G^*}"] \arrow[dr, "\Psi_G"'] & \\ & \mathfrak{Z}^{\mathrm{vst}}(G) \end{tikzcd}$$

which of course factor over the relevant subrings $\mathfrak{Z}^{\mathrm{FS}}$. According to a conjecture of Scholze-Shin [SS13, Remark 6.4], we expect that Ψ_G is always surjective and that Ψ_{G^*} is an isomorphism. In particular, we expect there is a unique surjective ring map $\mathfrak{Z}^{\mathrm{vst}}(G^*) \to \mathfrak{Z}^{\mathrm{vst}}(G)$ compatible with the diagram above. The following theorem gives an unconditional substitute for this map.

Theorem 1.4. *If $\Psi_{G^*}(f) = 0$, then $\Psi_G(f) = 0$. In other words, there is a unique surjective $\mathfrak{Z}^{\mathrm{spec}}(G^*)$-algebra map $\tau_G : \mathfrak{Z}^{\mathrm{FS}}(G^*) \to \mathfrak{Z}^{\mathrm{FS}}(G)$. This map enjoys the following compatibilities.*

i. *If $M \subset G$ is any Levi subgroup, with corresponding Levi $M^* \subset G^*$, the diagram*

$$\begin{tikzcd} \mathfrak{Z}^{\mathrm{FS}}(G^*) \arrow[r, "\tau_G"] \arrow[d] & \mathfrak{Z}^{\mathrm{FS}}(G) \arrow[d] \\ \mathfrak{Z}^{\mathrm{FS}}(M^*) \arrow[r, "\tau_M"] & \mathfrak{Z}^{\mathrm{FS}}(M) \end{tikzcd}$$

commutes.

ii. *The map τ_G is compatible with the usual transfer map $\mathrm{Trans}_G : SD^{\mathrm{temp}}(G^*) \to SD^{\mathrm{temp}}(G)$ on stable tempered virtual characters, in the sense that $\mathrm{Trans}_G(z \cdot \Theta) = \tau_G(z) \cdot \mathrm{Trans}_G(\Theta)$ for all $z \in \mathfrak{Z}^{\mathrm{FS}}(G^*)$ and all $\Theta \in SD^{\mathrm{temp}}(G^*)$.*

More generally, if f^* and f are any compactly supported functions on $G^*(E)$ and $G(E)$ with matching stable orbital integrals, then $z * f^*$ and $\tau_G(z) * f$ have matching stable orbital integrals.

Note that the existence of τ_G is not obviously related to stability, but our construction of this map crucially relies on Theorem 1.1. Part ii. is closely related to conjectures of Haines on "$\mathfrak{Z}$-transfer" for endoscopic groups [Hai14], and essentially confirms his conjectures in the special case of extended pure inner forms. It is surely true that Theorem 1.4 could be easily extended to all inner forms by some simple argument with z-extensions, but we have not attempted this.

The proofs of these results are not very long, but they involve several different flavors of mathematics, so let us briefly highlight the key ingredients. One basic idea is that Hecke operators acting on sheaves on Bun_G give rise to certain extra endomorphisms $\mathscr{T}_\mu$ of the space of virtual characters $D(G)$ which *commute* with the action of elements of $\mathfrak{Z}^{\mathrm{FS}}(G)$ (see Lemma 2.4). This is a decategorification of the well-known principle that Hecke operators and excursion operators commute on $D(\mathrm{Bun}_G, \overline{\mathbf{Q}_\ell})$. In principle, these endomorphisms could depend on our chosen isomorphism $\iota : \overline{\mathbf{Q}_\ell} \xrightarrow{\sim} \mathbf{C}$, but this dependence is actually harmless for our purposes.

We also adapt a wonderful idea from a recent paper of Chenji Fu [Fu24], who showed that Hecke eigensheaves on Bun_G at supercuspidal Fargues-Scholze parameters automatically give stable virtual characters at their stalks. The key observation here (recalled in a quantitative form in Section 2.2) is that the character formulas from [HKW22] show that as $\mu \to \infty$ in the appropriate sense, the limiting value of $\mathscr{T}_\mu$ on the regular elliptic set is a naive stable averaging.

Unfortunately, the character formulas in [HKW22] only give control over $\mathscr{T}_\mu$ on the regular elliptic set. The final key idea is to combine this control with Arthur's theory of elliptic tempered virtual characters [Art93, Art96]. The essential property of these gadgets is that they exactly span the subspace of tempered virtual characters which are fully controlled by their values on the regular elliptic set, and the complement of this subspace is spanned by parabolic inductions. Since the Fargues-Scholze map Ψ_G is compatible with parabolic induction, all together this gives precisely the right leverage to run arguments by induction on Levi subgroups. Although this aspect of Arthur's theory is certainly well-known in harmonic analysis, its use here in combination with the Fargues-Scholze machinery is new and seems to be very powerful. We will give some more applications of this technique elsewhere.

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Finally, I learned how to think about modern harmonic analysis on p-adic groups by reading Sandeep Varma's paper [Var24]. It's a particular pleasure to acknowledge the influence of this paper, and to thank Sandeep for answering my many questions.

2 Proofs

2.1 Harmonic analysis

In this section we collect some standard results in harmonic analysis. We learned essentially all of this material from [Var24]. Our notational conventions on Haar measures, Levis, Weyl groups, spaces of characters, etc. mostly follow Arthur's paper [Art96] and also coincide with the conventions in [Var24]. In particular, we fix a minimal Levi $M_0 \subset G$, with absolute Weyl group W_0, and then write $\mathcal{L}$ the set of standard Levis with its natural W_0-action, $W(M) = N_G(M)(E)/M(E)$ for any $M \in \mathcal{L}$, etc. We also fix once and for all a Haar measure dm on $M(E)$ for all $M \in \mathcal{L}$. We write $G(E)_{\mathrm{ell}}$ for the set of strongly regular elliptic elements.

Inside $C_c(G(E))$, we have the subspace $C_c(G(E))^{\mathrm{null}}$ of null functions f characterized by four equivalent conditions (see [Kaz86, Theorem 0], and see also [Dat00] for a nice discussion):

- $\mathrm{tr}(f|\pi) = 0$ for all irreducible representations π.
- $\mathrm{tr}(f|\pi) = 0$ for all tempered irreducible representations π.
- All regular semisimple orbital integrals of f are zero.

[2] More precisely, the crucial idea emerged in Frankfurt airport while waiting to board my flight home.

- f is in the subspace of commutators, i.e. the linear span of functions of the form $h(x) - h(gxg^{-1})$.

We write $\mathcal{I}(G) = C_c(G(E))/C_c(G(E))^{\text{null}}$. Note that the action of $\mathfrak{Z}(G)$ on $C_c(G(E))$ preserves null functions: if $f \in C_c(G(E))^{\text{null}}$ and $z \in \mathfrak{Z}(G)$, then $\text{tr}(z * f|\pi) = z_\pi \text{tr}(f|\pi) = 0$ for all irreducible π, so $z * f$ is null. Therefore the action descends to an action of $\mathfrak{Z}(G)$ on $\mathcal{I}(G)$ which we also denote as $z * f$.

For any Levi M, there is a canonical map $\mathfrak{Z}(G) \to \mathfrak{Z}(M)$, denoted $z \mapsto r_M(z)$ or just $z \mapsto r(z)$ if M is clear from context. There is also a canonical constant term map

$$\begin{aligned} &\mathcal{I}(G) \to \mathcal{I}(M) \\ &f \mapsto f_M = \delta_P^{1/2}(m) \int_{U(E)} \int_K f(kmuk^{-1}) dk du \end{aligned}$$

which strictly speaking is defined on $C_c(G(E))$, but it descends to the quotient $\mathcal{I}(G)$. Here $P = MU$ is any parabolic with Levi M, $K \subset G(E)$ is any open compact with dk the normalized Haar measure, and du is determined by our choices of Haar measures on $G(E)$ and $M(E)$. This map is characterized by the formula $\text{tr}(f_M|\pi) = \text{tr}(f|i_M^G \pi)$ for irreducible π. It is easy to see from this formula that $(z * f)_M = r_M(z) * f_M$.

We write $\mathcal{I}(G)^{\text{cusp}}$ for the subspace of *cuspidal* functions f characterized by the vanishing of f_M for all proper Levis M, or equivalently by the vanishing of all orbital integrals at non-elliptic regular semisimple elements. There is then a canonical decomposition

$$\mathcal{I}(G) = \bigoplus_{M \in \mathcal{L}/W_0} (\mathcal{I}(M)^{\text{cusp}})^{W(M)} \tag{1}$$

as recalled e.g. in [Var24, Section 4.2.1].

Dually, let $\text{Dist}(G)$ be the linear dual of $\mathcal{I}(G)$, so this is the space of all invariant distributions on G. Let $D(G) \subset \text{Dist}(G)$ be the subspace of virtual characters, and let $D^{\text{temp}}(G) \subset D(G)$ be the span of characters of tempered irreducible representations. The Bernstein center acts on $\text{Dist}(G)$ and preserves $D(G)$ and $D^{\text{temp}}(G)$. We write this action as $z \cdot \Theta$. This is compatible with the action on $\mathcal{I}(G)$ by the tautological formula $(z \cdot \Theta)(f) = \Theta(z * f)$. Of course, if $\Theta = \Theta_\pi$ is the character of an irreducible representation, then $z \cdot \Theta_\pi = z_\pi \Theta_\pi$.

Inside $D^{\text{temp}}(G)$, we have the still smaller subspace $D^{\text{ell}}(G)$ defined as the linear span of Arthur's elliptic tempered virtual characters $\Theta(\tau)$, $\tau \in T_{\text{ell}}(G)$ (see [Art93] for the notation). Then there is a canonical decomposition

$$D^{\text{temp}}(G) = \bigoplus_{M \in \mathcal{L}/W_0} (D^{\text{ell}}(M))^{W(M)} \tag{2}$$

where the inclusion of the M-indexed summand on the right-hand side is induced by the parabolic induction map $i_M^G : D(M) \to D(G)$. In particular, any $\Theta \in D^{\text{temp}}(G)$ admits a unique decomposition $\Theta = \Theta^{\text{ell}} + \Theta^{\text{ind}}$ where $\Theta^{\text{ell}} \in D^{\text{ell}}(G)$ and Θ^{ind} is in the span of the M-indexed summands for proper Levis M. We will freely and crucially use the fact that the pointwise evaluation map

$$\begin{aligned} D^{\text{ell}}(G) &\to C(G(E)_{\text{ell}}) \\ \Theta &\mapsto \Theta|_{G(E)_{\text{ell}}} \end{aligned}$$

is injective.

This decompositions (1) and (2) are perfectly dual to each other. In particular, any $f \in \mathcal{I}(G)$ admits a unique decomposition $f = f^{\text{cusp}} + f^{\text{nc}}$ such that f^{cusp} is cuspidal and $\Theta(f^{\text{nc}}) = 0$ for all $\Theta \in D^{\text{ell}}(G)$. Note that for any $\Theta \in D^{\text{temp}}(G)$, $\Theta(f) = \Theta^{\text{ell}}(f^{\text{cusp}}) + \Theta^{\text{ind}}(f^{\text{nc}})$.

Lemma 2.1. *For any $f \in \mathcal{I}(G)$ and $z \in \mathfrak{Z}(G)$, $(z * f)^{\text{cusp}} = z * f^{\text{cusp}}$ and $(z * f)^{\text{nc}} = z * f^{\text{nc}}$.*

Proof. As noted above, the action of $\mathfrak{Z}(G)$ on $\mathcal{I}(G)$ preserves cuspidal functions. On the other hand, the subset $D^{\text{ell}}(G) \subset D(G)$ is stable under the $\mathfrak{Z}(G)$-action, because any $z \in \mathfrak{Z}(G)$ acts on any $\Theta(\tau), \tau \in T_{\text{ell}}(G)$ through a scalar since all the irreducible characters occurring in a given $\Theta(\tau)$ have the same supercuspidal support. Therefore $\Theta(z * f^{\text{nc}}) = (z \cdot \Theta)(f^{\text{nc}}) = 0$ for all $\Theta \in D^{\text{ell}}(G)$, so $z * f^{\text{nc}}$ has vanishing cuspidal part. $\square$

All of the spaces of virtual characters defined above have stable analogues, denoted SD, SD^{temp}, SD^{ell}, etc. The decomposition of $D^{\text{temp}}(G)$ above admits a compatible stable analogue

$$SD^{\text{temp}}(G) = \bigoplus_{M \in \mathcal{L}/W_0} (SD^{\text{ell}}(M))^{W(M)}.$$

The elliptic inner product determines a canonical projection $D^{\text{ell}}(G) \to SD^{\text{ell}}(G)$ splitting the obvious inclusion, and the direct sum of these projections over $M \in \mathcal{L}/W_0$ yields an analogous projection $D^{\text{temp}}(G) \to SD^{\text{temp}}(G)$.

Recall that a function $f \in \mathcal{I}(G)$ is *unstable* if all its stable orbital integrals vanish. It is enough to impose this vanishing at strongly regular semisimple elements. We will need the result of Arthur that an element $f \in \mathcal{I}(G)$ is unstable iff $\Theta(f) = 0$ for all $\Theta \in SD^{\text{temp}}(G)$. For quasisplit groups this is explicitly proved in [Art96], and for general groups it is [Var24, Proposition 3.2.10]. This can be reformulated as follows.

Lemma 2.2. *The following conditions on an element $z \in \mathfrak{Z}(G)$ are equivalent.*

i. *For all unstable f, $z * f$ is unstable.*

ii. *The endomorphism $z \cdot$ of $D(G)$ preserves $SD(G)$.*

iii. *The endomorphism $z \cdot$ of $D^{\text{temp}}(G)$ preserves $SD^{\text{temp}}(G)$.*

Proof. Clearly i. implies ii. implies iii. That iii. implies i. is exactly the result of Arthur quoted before the lemma. $\square$

As in the introduction, we call elements of the Bernstein center satisfying these equivalent conditions *very stable*. By Kazhdan's density theorem, it is easy to see that $D^{\text{temp}}(G)$ is a faithful $\mathfrak{Z}(G)$-module. This result has an easy stable analogue.

Lemma 2.3. *Under the natural action, $SD^{\text{temp}}(G)$ is a faithful $\mathfrak{Z}^{\text{vst}}(G)$-module.*

Proof. Let $z \in \mathfrak{Z}^{\text{vst}}(G)$ be an element such that $z \cdot \Theta = 0$ for all stable tempered virtual characters Θ. Then $\Theta(z * f) = 0$ for all f and all such Θ, so by Arthur's result recalled above, $z * f$ has vanishing stable orbital integrals for all f. Now the invariant distribution $\delta : h \to h(1)$ is stable [Kot88, Proposition 1], i.e. it is in the closed linear span of stable orbital integrals, so $0 = \delta(z * f) = z(f)$ for all f. Therefore $z = 0$ as desired. $\square$

2.2 Excursion versus Hecke

The key extra symmetry of elements $z \in \mathfrak{Z}^{\text{FS}}(G)$ which will enforce their stability is their commutation with certain endomorphisms of $D(G)$ coming from Hecke operators on Bun_G. More formally, let $\mu : \mathbf{G}_{m,\overline{E}} \to G_{\overline{E}}$ be a conjugacy class of cocharacters such that $1 \in B(G, \mu)$, or equivalently such that $V_\mu|_{Z(\hat{G})^\Gamma}$ is trivial, where V_μ is the irreducible representation of $\hat{G}$ with highest weight μ. Then $i_1^* T_{V_\mu^*} i_{1!}$ defines an endofunctor on the derived category of smooth $G(E)$-representations with

$\overline{\mathbf{Q}_\ell}$-coefficients, which preserves the finite length objects and hence induces an endomorphism on the Grothendieck group $K_0\mathrm{Rep}_{\mathrm{fl}}(G(E), \overline{\mathbf{Q}_\ell})$. Transporting this endomorphism across our fixed isomorphism ι and $\mathbf{C}$-linearizing, we get an endomorphism $\mathscr{T}_\mu : D(G) \to D(G)$.[3] By [HKW22, Theorem 6.5.4], there is a (necessarily unique) linear map $t_\mu : \mathcal{I}(G) \to \mathcal{I}(G)$ such that $\mathscr{T}_\mu(\Theta)(f) = \Theta(t_\mu(f))$ for all f.

Lemma 2.4. *For any z in $\mathfrak{Z}^{\mathrm{FS}}(G)$, we have $z * t_\mu(f) = t_\mu(z * f)$. Equivalently, $z\cdot$ and $\mathscr{T}_\mu$ commute as endomorphisms of $D(G)$.*

This commutation of $z\cdot$ and $\mathscr{T}_\mu$ is the crucial extra symmetry we will exploit.

Proof. Fix any irreducible representation π, with $z \cdot \pi = z_\pi \pi$. Write $\mathscr{T}_\mu(\Theta_\pi) = \sum n_i \Theta_{\pi_i}$. Since Hecke operators and excursion operators commute, $z \cdot \pi_i = z_\pi \pi_i$ for all i. Then

$$\begin{aligned}
\Theta_\pi\left(t_\mu(z * f)\right) &= \mathscr{T}_\mu(\Theta_\pi)(z * f) \\
&= (z \cdot \mathscr{T}_\mu(\Theta_\pi))(f) \\
&= z_\pi \mathscr{T}_\mu(\Theta_\pi)(f) \\
&= z_\pi \Theta_\pi(t_\mu(f)) \\
&= (z \cdot \Theta_\pi)(t_\mu(f)) \\
&= \Theta_\pi\left(z * t_\mu(f)\right).
\end{aligned}$$

Therefore $z * t_\mu(f) - t_\mu(z * f)$ has trace zero on all irreducible representations, so it vanishes. ☐

We will also need some very non-formal facts about the operator $\mathscr{T}_\mu$. These all follow from the main results of [HKW22], which give an explicit formula for the restriction of $\mathscr{T}_\mu(\Theta)$ to $G(E)_{\mathrm{ell}}$ for any Θ. We now recall this formula. Fix any $g \in G(E)_{\mathrm{ell}}$ with centralizer T_g, and let $[[g]]$ denote the set of conjugacy classes in the stable conjugacy class of g. For any element $g' \in [[g]]$, we defined a certain invariant $\mathrm{inv}(g, g') \in B(T_g) = X_*(T_g)_\Gamma$. This invariant has the property that for each $\lambda \in X_*(T_g)$ such that $\mathrm{dim} V_\mu[\lambda] \neq 0$, there is *exactly* one element $g' \in [[g]]$ such that $\overline{\lambda} = \mathrm{inv}(g, g')$ in $X_*(T_g)_\Gamma$. Here $\overline{\lambda}$ is the natural projection of λ along $X_*(T_g) \to X_*(T_g)_\Gamma$. In this notation, the character formula proved in [HKW22, Theorem 6.5.2] says that

$$\mathscr{T}_\mu(\Theta)(g) = \sum_{\substack{\lambda \in X_*(T_g),\, g' \in [[g]] \\ \mathrm{inv}(g,g') = \overline{\lambda}}} \mathrm{dim} V_\mu[\lambda] \cdot \Theta(g')$$

for any $\Theta \in D(G)$.

We record a few consequences of this result.

Proposition 2.5. *Fix μ as above.*
i. *If f is cuspidal, then $t_\mu(f)$ is cuspidal.*
ii. *If $\Theta \in D(G)$ is stable, then $\mathscr{T}_\mu(\Theta) = \mathrm{dim} V_\mu \cdot \Theta + \Theta'$ where Θ' is parabolically induced.*

Proof. In general, $\Theta \in D(G)$ is parabolically induced if and only if $\Theta|_{G(E)_{\mathrm{ell}}}$ vanishes identically (see [HKW22, Appendix C]).

[3] This endomorphism depends on our fixed isomorphism $\iota : \mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}_\ell}$, which we suppress from the notation. Note however that by the main results of [HKW22] recalled below, the restriction of $\mathscr{T}_\mu(\Theta)$ to $G(E)_{\mathrm{ell}}$ is completely independent of this choice.

Part i. is dual to the fact that if Θ is parabolically induced, then also $\mathscr{T}_\mu(\Theta)$ is parabolically induced. This follows from the formula for the character of $\mathscr{T}_\mu(\Theta)$ at elliptic elements recalled above, since if $\Theta|_{G(E)_{\mathrm{ell}}} = 0$ then visibly $\mathscr{T}_\mu(\Theta)|_{G(E)_{\mathrm{ell}}} = 0$. Part ii. follows similarly, since if Θ is stable then

$$\begin{aligned}\mathscr{T}_\mu(\Theta)(g) &= \sum_{\substack{\lambda\in X_*(T_g),\, g'\in[[g]] \\ \mathrm{inv}(g,g')=\overline{\lambda}}} \dim V_\mu[\lambda]\cdot\Theta(g') \\ &= \sum_{\lambda\in X_*(T_g)} \dim V_\mu[\lambda]\cdot\Theta(g) \\ &= \dim V_\mu\cdot\Theta(g)\end{aligned}$$

for all $g\in G(E)_{\mathrm{ell}}$, so $\mathscr{T}_\mu(\Theta) - \dim V_\mu\cdot\Theta$ vanishes identically on $G(E)_{\mathrm{ell}}$. □

We will also adapt a marvelous idea from Chenji Fu's paper [Fu24], showing that as $\mu\to\infty$ in an appropriate sense, $\mathscr{T}_\mu$ implements a *stable averaging*. We will need a version of this result with some uniformity in g. To explain this, note that for any given $g\in G(E)_{\mathrm{ell}}$, we can rearrange the character formula as

$$\mathscr{T}_\mu(\Theta)(g) = \sum_{g'\in[[g]]}\Theta(g')\sum_{\substack{\lambda\in X_*(T_g) \\ \mathrm{inv}(g,g')=\overline{\lambda}}}\dim V_\mu[\lambda].$$

Now set $\mu_m = 4m\rho_G$ for $m\geq 1$, and consider the rational number

$$C_m(g,g') = \frac{\sum_{\substack{\lambda\in X_*(T_g) \\ \mathrm{inv}(g,g')=\overline{\lambda}}}\dim V_{\mu_m}[\lambda]}{\dim V_{\mu_m}},$$

so we trivially get

$$\frac{1}{\dim V_{\mu_m}}\mathscr{T}_{\mu_m}(\Theta)(g) = \sum_{g'\in[[g]]}\Theta(g')C_m(g,g').$$

On the other hand, Fu's analysis shows that for sufficiently large m, we have $|C_m(g,g') - \frac{1}{|[[g]]|}| \leq \frac{C}{m}$ for some fixed constant C which depends only on the ambient group G. Now for any function $\phi\in C(G(E)_{\mathrm{ell}}//G(E))$, define its stable average by the formula $\phi_{\mathrm{st}}(g) = \frac{1}{|[[g]]|}\sum_{g'\in[[g]]}\phi(g')$. Then we get that $\frac{1}{\dim V_{\mu_m}}\mathscr{T}_{\mu_m}(\Theta)(g)\to\Theta_{\mathrm{st}}(g)$ pointwise on $G(E)_{\mathrm{ell}}$, and in fact that

$$|\frac{1}{\dim V_{\mu_m}}\mathscr{T}_{\mu_m}(\Theta)(g) - \Theta_{\mathrm{st}}(g)| \leq \frac{C}{m}\mathrm{sup}_{x\in[[g]]}|\Theta(x)|$$

for all $g\in G(E)_{\mathrm{ell}}$ where C depends only on G. Since the Weyl discriminant is invariant under stable conjugacy, we can insert it into the above estimate, giving

$$|D(g)|^{1/2}|\frac{1}{\dim V_{\mu_m}}\mathscr{T}_{\mu_m}(\Theta)(g) - \Theta_{\mathrm{st}}(g)| \leq \frac{C}{m}\mathrm{sup}_{x\in[[g]]}|D(x)|^{1/2}|\Theta(x)|.$$

Now by a deep theorem of Harish-Chandra, $|D(x)|^{1/2}|\Theta(x)|$ (extended by zero from $G(E)_{\mathrm{reg.ss}}$ to $G(E)$) is bounded on any compact subset of $G(E)$. (See e.g. [Clo87] for a proof of a more general result.) Putting things together, we deduce in particular that if $U\subset G(E)$ is any compact subset whose elliptic part is stably invariant in the weak sense that for all $g \in G(E)_{\mathrm{ell}}$ either $[[g]] \cap U = 0$ or U meets *every* conjugacy class in $[[g]]$, then

$$|D(g)|^{1/2} \left| \frac{1}{\dim V_{\mu_m}} \mathscr{T}_{\mu_m}(\Theta)(g) - \Theta_{\mathrm{st}}(g) \right|$$

tends to zero *uniformly* for $g \in G(E)_{\mathrm{ell}} \cap U$ as $m \to \infty$.

2.3 Stability

In this section we prove Theorem 1.1.

Let $z \in \mathfrak{Z}(G)$ be in the image of the Fargues-Scholze map Ψ_G. We need to prove that for any unstable $f \in \mathcal{I}(G)$, $z * f$ is unstable. For this, it is enough to see that $\Theta(z * f) = 0$ for all $\Theta \in SD^{\mathrm{temp}}(G)$ as recalled in Section 2.1. We will prove this by induction on the semisimple rank of G.

First suppose Θ is parabolically induced. Without loss of generality we can assume $\Theta = i_M^G \Theta_M$ for some $\Theta_M \in SD^{\mathrm{temp}}(M)$ and some proper Levi M. Then

$$\Theta(z * f) = \Theta_M((z * f)_M) = \Theta_M(r_M(z) * f_M).$$

Since f is unstable, also f_M is unstable. Now $r_M(z)$ is in the image of Ψ_M by compatiblity of the Fargues-Scholze map with parabolic induction, so by induction on the semisimple rank we know that $r_M(z) * f_M$ is unstable, and thus $\Theta_M(r_M(z) * f_M) = 0$.

This reduces us to the case where $\Theta \in SD^{\mathrm{ell}}(G)$. By Lemma 2.1, we can assume our unstable function f is cuspidal, in which case also $z * f$ is cuspidal. Now, with μ as in Section 2.2, consider the quantity

$$C_\mu := \frac{1}{\dim V_\mu} \mathscr{T}_\mu(\Theta)(z * f)$$

By Proposition 2.5, $\mathscr{T}_\mu(\Theta) = \dim V_\mu \cdot \Theta + \Theta'$ for some parabolically induced Θ'. Since $z * f$ is cuspidal, $\Theta'(z * f) = 0$, so this simplifies to $C_\mu = \Theta(z * f)$ which is evidently a constant independent of μ. Our goal is to show that this constant vanishes. Writing $\Xi = z \cdot \Theta$, Lemma 2.4 shows that

$$\begin{aligned} C_\mu &= \frac{1}{\dim V_\mu} (z \cdot \mathscr{T}_\mu(\Theta))(f) \\ &= \frac{1}{\dim V_\mu} \mathscr{T}_\mu(z \cdot \Theta)(f) \\ &= \frac{1}{\dim V_\mu} \mathscr{T}_\mu(\Xi)(f) \end{aligned}$$

for any μ. Note that although Θ is stable, Ξ certainly need not be stable a priori.[4]

At this point we use Fu's method. More precisely, taking $\mu = \mu_m$ with $m \to \infty$ as in Section 2.2, we will use that the operator $\frac{1}{\dim V_\mu} \mathscr{T}_\mu(\Xi)$ effects a stable averaging as discussed there. To implement this, for any $\Theta \in D^{\mathrm{temp}}(G)$, let $\Theta^{\mathrm{st}} \in SD^{\mathrm{temp}}(G)$ be its stable projection. Writing $\frac{1}{\dim V_{\mu_m}} \mathscr{T}_{\mu_m}(\Xi) = \Xi^{\mathrm{st}} + \Phi_{\mu_m}$, it is clear that $\Xi^{\mathrm{st}}(f) = 0$ since f is unstable, so $C_{\mu_m} = \Phi_{\mu_m}(f)$. We will now show that as $m \to \infty$, $\Phi_{\mu_m}(f) \to 0$.

To proceed further, we exploit the cuspidality of f to rewrite $\Phi_{\mu_m}(f)$ via a simple form the Weyl integration formula. More precisely, fix a Haar measure da on the split center $A_G(E)$, and set

[4] In fact, by Lemma 2.2, we are exactly trying to prove that $z \cdot$ preserves stability.

$O_\gamma(f) = \int_{A_G(E)\backslash G(E)} f(x^{-1}\gamma x)dx$ as a function on $G(E)_{\mathrm{ell}}$, where $dx = dg/da$ in the usual manner. Then for any $\Theta \in D(G)$ and any $f \in \mathcal{I}(G)^{\mathrm{cusp}}$, the Weyl integration formula can be written as

$$\Theta(f) = \sum_T \frac{1}{|W(G,T)(E)|} \int_{T(E)} \Theta(t) O_t(f) |D(t)| dt.$$

Here the sum runs over a (finite) set of representatives for the $G(E)$-conjugacy classes of elliptic maximal tori in G, and dt is the Haar measure on $T(E)$ determined by the chosen Haar measure on $A_G(E)$ and the normalized Haar measure on the compact group $T(E)/A_G(E)$. We briefly recall some facts about convergence. For each T, the set of elements $t \in T(E) \cap G(E)_{\mathrm{ell}}$ such that $O_t(f) \neq 0$ has compact closure C_T in $T(E)$. Now, by fundamental results of Harish-Chandra, the function $|D(g)|^{1/2}\Theta(g)$ (extended by zero from the regular semisimple locus) is locally bounded on $G(E)$ (as recalled in Section 2.2), and $|D(\gamma)|^{1/2} O_\gamma(f)$ is a bounded function on $G(E)_{\mathrm{ell}}$ (see e.g. [Art91, Section 4]). In particular, for a fixed cuspidal f and varying Θ, we can replace each integral above by an integral over the fixed compact subset $C_T \subset T(E)$, and the integrand is a bounded function on that compact subset and is locally constant on a dense open subset thereof.

Now substituting in Φ_{μ_m} for Θ in the Weyl integration formula above, we are reduced to showing that $|D(t)|^{1/2}\Phi_{\mu_m}(t) \to 0$ uniformly on C_T as $m \to \infty$. Here again, $|D(t)|^{1/2}\Phi_{\mu_m}(t)$ is defined a priori as a bounded function on $C_T \cap G(E)_{\mathrm{ell}}$ and extended by zero to C_T. Recall that by definition, $\Phi_{\mu_m} = \frac{1}{\mathrm{dim} V_{\mu_m}} \mathscr{T}_{\mu_m}(\Xi) - \Xi^{\mathrm{st}}$. First we compute the restriction of Ξ^{st} to $G(E)_{\mathrm{ell}}$. This follows from some general theory: by [Var24, Lemma 3.4.5], $\Theta^{\mathrm{st}}|_{G(E)_{\mathrm{ell}}} = (\Theta|_{G(E)_{\mathrm{ell}}})_{\mathrm{st}}$ for any $\Theta \in D(G)$, where $f \mapsto f_{\mathrm{st}}$ is the naive stable averaging discussed in Section 2.2. In particular, this applies to Ξ, so we get

$$\Phi_{\mu_m}(g) = \frac{1}{\mathrm{dim} V_{\mu_m}} \mathscr{T}_{\mu_m}(\Xi)(g) - \Xi_{\mathrm{st}}(g)$$

for any $g \in G(E)_{\mathrm{ell}}$. Now choose a compact subset $U \subset G(E)_{\mathrm{ell}}$ as in the discussion at the end of Section 2.2 which moreover contains C_T for each T. Then by the discussion there, $|D(g)|^{1/2}\Phi_{\mu_m}(g) \to 0$ uniformly in m for all $g \in U \cap G(E)_{\mathrm{ell}}$, and in particular for all $t \in C_T \cap G(E)_{\mathrm{ell}}$. But then this immediately extends to the same statement for all $t \in C_T$ since $|D(g)|^{1/2}\Phi_{\mu_m}(g)$ is extended by zero from the regular semisimple part.

Putting all of this together, we get that

$$\Phi_{\mu_m}(f) = \sum_T \frac{1}{|W(G,T)(E)|} \int_{C_T} |D(t)|^{1/2}\Phi_{\mu_m}(t) \cdot |D(t)|^{1/2} O_t(f) dt$$

where $C_T \subset T(E)$ is compact, both halves of the integrand are bounded on C_T, and $|D(t)|^{1/2}\Phi_{\mu_m}(t) \to 0$ *uniformly* on C_T for each T as $m \to \infty$. Therefore $\Phi_{\mu_m}(f) \to 0$ as $m \to \infty$. This gives the result.

2.4 Inner forms

In this section we deal with Theorem 1.4. To construct the map τ_G, we proceed by induction on the semisimple rank. More precisely, fix some element $f \in \mathfrak{Z}^{\mathrm{spec}}(G^*)$ such that $z := \Psi_G(f) \neq 0$. We need to show that $z^* := \Psi_{G^*}(f) \neq 0$.

By Lemma 2.3, $SD^{\mathrm{temp}}(G)$ is a faithful $\mathfrak{Z}^{\mathrm{vst}}(G)$-module, thus a faithful $\mathfrak{Z}^{\mathrm{FS}}(G)$-module by Theorem 1.1. In particular, the endomorphism $z\cdot$ of $SD^{\mathrm{temp}}(G)$ is not identically zero. Suppose first that $z \cdot \Theta \neq 0$ for some parabolically induced $\Theta = i_M^G \Theta_M$ with $\Theta_M \in SD^{\mathrm{temp}}(M)$. Let M^* be the

Levi subgroup corresponding to M. We have a commutative diagram

$$\begin{tikzcd}
 & & \mathfrak{Z}^{\mathrm{FS}}(G) \arrow[dd, "r"] \\
\mathfrak{Z}^{\mathrm{spec}}(G^*) \arrow[r, "\Psi_{G^*}"'] \arrow[d, "r^{\mathrm{spec}}"] \arrow[urr, "\Psi_G"] & \mathfrak{Z}^{\mathrm{FS}}(G^*) \arrow[d, "r^*"] & \\
\mathfrak{Z}^{\mathrm{spec}}(M^*) \arrow[r, "\Psi_{M^*}"] \arrow[drr, "\Psi_M"'] & \mathfrak{Z}^{\mathrm{FS}}(M^*) \arrow[dr, "\tau_M"] & \\
 & & \mathfrak{Z}^{\mathrm{FS}}(M)
\end{tikzcd}$$

where τ_M exists and is surjective by induction on the semisimple rank. Now by basic properties of the Bernstein center, the linear map $z \cdot i_M^G(-)$ coincides with the linear map $i_M^G(r(z) \cdot -)$. In particular, since $z \cdot i_M^G \Theta_M \neq 0$, we get that $r(z) \neq 0$. But then

$$\begin{aligned}
r(z) &= \Psi_M(r^{\mathrm{spec}}(f)) \\
&= \tau_M(\Psi_{M^*}(r^{\mathrm{spec}}(f))) \\
&= \tau_M(r^*(\Psi_{G^*}(f))) \\
&= \tau_M(r^*(z^*))
\end{aligned}$$

using the commutativity of the diagram, so $z^* \neq 0$.

It remains to deal with the case where z annihilates all parabolically induced elements of $SD^{\mathrm{temp}}(G)$. By Lemma 2.3, we may choose some $\Theta \in SD^{\mathrm{ell}}(G)$ such that $z \cdot \Theta \neq 0$. Pick some μ as in Section 2.2 such that $b \in B(G^*, \mu)$, and let $\mathscr{T}_\mu : D(G) \to D(G^*)$ be the linear map induced by $i_1^* T_{V_\mu^*} i_{b!}$ (and our choice of isomorphism $\mathbf{C} \simeq \overline{\mathbf{Q}_\ell}$) as in the discussion there. By the commutation of Hecke operators with excursion operators, we get that $z^* \cdot \mathscr{T}_\mu(\Theta) = \mathscr{T}_\mu(z \cdot \Theta)$ as in Lemma 2.4. By assumption, $z \cdot \Theta \in SD^{\mathrm{ell}}(G)$ is nonzero, so it is not identically zero on $G(E)_{\mathrm{ell}}$. Now the character formula [HKW22, Theorem 6.5.2] again shows that for all matching stably conjugate pairs $G^*(E)_{\mathrm{ell}} \ni g^* \sim^{\mathrm{st}} g \in G(E)_{\mathrm{ell}}$,

$$\mathscr{T}_\mu(z \cdot \Theta)(g^*) = e(G) \mathrm{dim} V_\mu (z \cdot \Theta)(g),$$

so $\mathscr{T}_\mu(z \cdot \Theta)(g^*)$ is nonzero for some $g^* \in G^*(E)_{\mathrm{ell}}$. Therefore $\mathscr{T}_\mu(z \cdot \Theta) = z^* \cdot \mathscr{T}_\mu(\Theta) \neq 0$, so $z^* \neq 0$ as desired.

Next, for the compatibility with parabolic induction, note that we already have a diagram

$$\begin{tikzcd}
 & & \mathfrak{z}^{\mathrm{FS}}(G) \arrow[ddd, "r"] \\
\mathfrak{z}^{\mathrm{spec}}(G^*) \arrow[r, "\Psi_{G^*}"'] \arrow[urr, "\Psi_G"] \arrow[d, "r^{\mathrm{spec}}"] & \mathfrak{z}^{\mathrm{FS}}(G^*) \arrow[ur, "\tau_G"'] \arrow[d, "r^*"] & \\
\mathfrak{z}^{\mathrm{spec}}(M^*) \arrow[r, "\Psi_{M^*}"] \arrow[drr, "\Psi_M"'] & \mathfrak{z}^{\mathrm{FS}}(M^*) \arrow[dr, "\tau_M"] & \\
 & & \mathfrak{z}^{\mathrm{FS}}(M)
\end{tikzcd}$$

where everything commutes except possibly the trapezoid spanned by r, r^*, τ_G, τ_M. But the surjectivity of Ψ_{G^*} immediately implies that this trapezoid commutes as well.

It remains to show compatibility with the transfer map. We first recall some properties of this map, referring to [Var24, Section 3.2] for details.[5] Fixing an inner twist and all other data as in [Var24, Section 3.2], we get a canonical injection $\mathcal{L}/W_0 \to \mathcal{L}^*/W_0^*$. The transfer map $\mathrm{Trans}_G : SD^{\mathrm{temp}}(G^*) \to SD^{\mathrm{temp}}(G)$ is then compatible with the grading

$$SD^{\mathrm{temp}}(G^*) = \bigoplus_{M^* \in \mathcal{L}^*/W_0^*} (SD^{\mathrm{ell}}(M^*))^{W(M^*)}$$

and its analogue for G in the following very strong sense.

- Its restriction to the summand $SD^{\mathrm{ell}}(G^*)$ factors over an isomorphism $\mathrm{Trans}_G^{\mathrm{ell}} : SD^{\mathrm{ell}}(G^*) \xrightarrow{\sim} SD^{\mathrm{ell}}(G)$ characterized by the equality $\mathrm{Trans}_G^{\mathrm{ell}}(\Theta)(g) = e(G)\Theta(g^*)$ for all matching stably conjugate pairs $G^*(E)_{\mathrm{ell}} \ni g^* \sim^{\mathrm{st}} g \in G(E)_{\mathrm{ell}}$.
- If $M^* \in \mathcal{L}^*/W_0^*$ is irrelevant in the sense that it is not the image of some $M \in \mathcal{L}/W_0$, Trans_G is identically zero on the M^*-indexed summand.
- If M^* is the image of some M, then $\mathrm{Trans}_G i_{M^*}^{G^*}\Theta_{M^*} = i_M^G \mathrm{Trans}_M^{\mathrm{ell}}\Theta_{M^*}$ for all $\Theta_{M^*} \in (SD^{\mathrm{ell}}(M^*))^{W(M^*)}$, compatibly with the Weyl equivariance via the appropriate identification $W(M) = W(M^*)$.

Now let $z \in \mathfrak{z}^{\mathrm{FS}}(G^*)$ be any element, and pick any $\Theta \in SD^{\mathrm{temp}}(G^*)$. We need to show that $\mathrm{Trans}_G(z \cdot \Theta) = \tau_G(z) \cdot \mathrm{Trans}_G(\Theta)$. First suppose Θ is parabolically induced, say of the form $i_{M^*}^{G^*}\Xi$ for some $\Xi \in SD^{\mathrm{ell}}(M^*)$. Then by the remarks on the grading above and compatibility of the Bernstein center action with parabolic induction, we compute that

$$\mathrm{Trans}_G(z \cdot i_{M^*}^{G^*}\Xi) = \mathrm{Trans}_G(i_{M^*}^{G^*}(r^*(z) \cdot \Xi)).$$

If M^* is irrelevant, this is identically zero, as is $\mathrm{Trans}_G(\Theta)$, so there is nothing to prove. If M^* is

[5] Our convention differs from Varma's in one place only: we normalize the transfer factor between G and G^* to be the Kottwitz sign $e(G)$, rather than the scalar 1 as in [Var24].

relevant, we compute further that

$$\begin{aligned}\mathrm{Trans}_G(i_{M^*}^{G^*}(r^*(z)\cdot\Xi)) &= i_M^G\mathrm{Trans}_M^{\mathrm{ell}}(r^*(z)\cdot\Xi)\\ &= i_M^G\tau_M(r^*(z))\cdot\mathrm{Trans}_M^{\mathrm{ell}}\Xi\\ &= i_M^G r(\tau_G(z))\cdot\mathrm{Trans}_M^{\mathrm{ell}}\Xi\\ &= \tau_G(z)\cdot i_M^G\mathrm{Trans}_M^{\mathrm{ell}}\Xi\\ &= \tau_G(z)\cdot\mathrm{Trans}_G(\Theta),\end{aligned}$$

where the second equality follows by induction on the semisimple rank.

This reduces us to the case that $\Theta\in SD^{\mathrm{ell}}(G^*)$. Here we just need to see that $(\tau(z)\cdot\mathrm{Trans}_G^{\mathrm{ell}}(\Theta))(g)=\mathrm{Trans}_G^{\mathrm{ell}}(z\cdot\Theta)(g)$ for all $g\in G(E)_{\mathrm{ell}}$. Now we exploit [HKW22] one more time. More precisely, defining $\mathscr{T}_\mu^*:D(G^*)\to D(G)$ as the map induced by $i_b^*T_{V_\mu}i_{i!}$ (with b and μ as above), the character formula [HKW22, Theorem 6.5.2] shows that

$$\mathscr{T}_\mu^*(\Theta)(g)=\dim V_\mu\cdot\mathrm{Trans}_G^{\mathrm{ell}}(\Theta)(g)$$

for all $g\in G(E)_{\mathrm{ell}}$. But again, Hecke operators commute with excursion operators, so we get that $\mathscr{T}_\mu^*(z\cdot\Theta)=\tau_G(z)\cdot\mathscr{T}_\mu^*(\Theta)$. Evaluating both sides of this equality on any $g\in G(E)_{\mathrm{ell}}$ and invoking the character formula, we get the result.

Finally, suppose f^* and f are any matching functions. Fix any $z\in\mathfrak{Z}^{\mathrm{FS}}(G^*)$, and let $(\tau_G(z)*f)^*$ be a function on $G^*(E)$ matching $\tau_G(z)*f$. We need to see that $h:=z*f^*-(\tau_G(z)*f)^*$ is unstable. For this, pick any $\Theta\in SD^{\mathrm{temp}}(G^*)$. We then compute

$$\begin{aligned}\Theta(z*f^*) &= (z\cdot\Theta)(f^*)\\ &= \mathrm{Trans}_G(z\cdot\Theta)(f)\\ &= (\tau_G(z)\cdot\mathrm{Trans}_G\Theta)(f)\\ &= \mathrm{Trans}_G(\Theta)(\tau_G(z)*f)\\ &= \Theta\left((\tau_G(z)*f)^*\right)\end{aligned}$$

where the first and fourth equalities are trivial, the second and fifth equalities follow from the definition of the transfer map, and the third equality follows from our results so far. Therefore $\Theta(h)=0$, and since $\Theta\in SD^{\mathrm{temp}}(G^*)$ is arbitrary, this implies that h is unstable, as desired.

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