Excursion operators and the stable Bernstein center

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Abstract

We prove that the Fargues-Scholze construction of elements in the Bernstein center via excursion operators always yields stable distributions. We also prove a strong quantitative compatibility of the Fargues-Scholze construction with transfer across extended pure inner forms. The proofs combine the character formulas from [HKW22], the commutation of Hecke operators with excursion operators, an averaging trick due to Fu [Fu24], and Arthur's theory of elliptic tempered virtual characters. The arguments work uniformly for all connected reductive groups over p-adic local fields.

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1 Introduction

1.1 Main results

Fix a finite extension $E/\mathbf{Q}_p$, and let G/E be a connected reductive group. Let $\mathfrak{Z}(G)$ be the Bernstein center of G, regarded as the convolution algebra of essentially compact invariant distributions on $G(E)$. This acts by convolution on the space of compactly supported continuous functions $C_c(G(E), \mathbf{C})$, and on all smooth complex $G(E)$-modules via the identification of $\mathfrak{Z}(G)$ with the center of the category. If π is a smooth irreducible $G(E)$-representation, we write $z_\pi \in \mathbf{C}$ for the scalar with $z \cdot \pi = z_\pi \pi$. For the basic theory of the Bernstein center, the reader should consult [Ber84], and excellent overviews can also be found in [BDK86, Hai14].

Following the usual terminology, we say a distribution $z \in \mathfrak{Z}(G)$ is *stable* if it vanishes on unstable functions f, i.e. on functions with vanishing stable orbital integrals. These form a submodule $\mathfrak{Z}^{\mathrm{st}}(G) \subset \mathfrak{Z}(G)$. We say a distribution z is *very stable* if $z * f$ is unstable for all unstable f. Although this condition (first singled out by Scholze-Shin [SS13, Conjecture 6.3]) is a priori more restrictive than stability, in practice it is much easier to verify. It is easy to see that very stable distributions are stable, and that they form a commutative subalgebra $\mathfrak{Z}^{\mathrm{vst}}(G) \subset \mathfrak{Z}(G)$ such that $\mathfrak{Z}^{\mathrm{st}}(G)$ is naturally a $\mathfrak{Z}^{\mathrm{vst}}(G)$-module.

These stability conditions are expected to play a key role in local harmonic analysis and the local Langlands correspondence. Indeed, it is a by-now-standard conjecture that stable and very stable elements of $\mathfrak{Z}(G)$ coincide (this is [Var24, Conjecture 1.1.4]), and that z is (very) stable if and only if for all tempered L-packets $\Pi_\phi(G)$, $z_{\pi_1} = z_{\pi_2}$ for all $\pi_1, \pi_2 \in \Pi_\phi(G)$ (this is formulated in [BKV15], for instance). For groups with a sufficiently well-understood local Langlands correspondence, this conjecture was recently proved by Varma in a beautiful paper [Var24, Theorem 4.4.2]. However, the second part of this conjecture certainly doesn't make any sense without prior knowledge of the local Langlands correspondence, and it seems extremely hard to construct (very) stable central elements from scratch (but see [BKV15, BKV16] for some interesting results in this direction). The importance of constructing elements of the stable center for global purposes has previously been emphasized by Haines [Hai14], who also highlighted the expected connection with algebraic functions on the variety of semisimple L-parameters.

In this paper we show that the Fargues-Scholze machinery is perfectly suited to the construction of very stable central distributions. More precisely, let $\mathfrak{Z}^{\mathrm{spec}}(G)$ be the ring of global functions on the variety of semisimple L-parameters for G. In their amazing paper [FS24], Fargues-Scholze constructed a canonical ring map $\Psi_G : \mathfrak{Z}^{\mathrm{spec}}(G) \to \mathfrak{Z}(G)$ satisfying a long list of compatibilities, using V. Lafforgue's formalism of excursion operators [Laf18] adapted to the Fargues-Fontaine curve.[1] To streamline the discussion, let us write $\mathfrak{Z}^{\mathrm{FS}}(G)$ for the image of Ψ_G. Our first main result is the following theorem, essentially confirming a conjecture of Haines [Hai14] and Scholze-Shin [SS13, Conjecture 6.3].

Theorem 1.1. *The map $\Psi_G : \mathfrak{Z}^{\mathrm{spec}}(G) \to \mathfrak{Z}(G)$ factors over the subalgebra of very stable central distributions. Equivalently, there is an inclusion $\mathfrak{Z}^{\mathrm{FS}}(G) \subseteq \mathfrak{Z}^{\mathrm{vst}}(G)$.*

We emphasize that G is completely arbitrary. While we expect the inclusion $\mathfrak{Z}^{\mathrm{FS}}(G) \subseteq \mathfrak{Z}^{\mathrm{vst}}(G)$ is an equality for all groups, this seems far out of reach.

This theorem has several corollaries. First, recall that a virtual character $\Theta = \sum_{1 \le i \le j} a_i \Theta_{\pi_i}$ is *atomically stable* if Θ is stable, with all coefficients $a_i \neq 0$, and no smaller linear combination $\sum_{i \in I \subsetneq [1,j]} b_i \Theta_{\pi_i}$ is stable.

Corollary 1.2. *If $\Theta = \sum a_i \Theta_{\pi_i}$ is an atomically stable virtual character, the Fargues-Scholze parameter φ_{π_i} is independent of i.*

Now suppose G splits over a tame extension and $p \nmid |W_G|$. Then for any regular supercuspidal parameter $\phi : W_E \to {}^L G$, Kaletha [Kal19] explicitly constructed a supercuspidal L-packet $\Pi_\phi(G)$. By work of Fintzen-Kaletha-Spice [FKS23], the linear combination $S\Theta_\phi = \sum_{\pi \in \Pi_\phi(G)} \Theta_\pi$ is atomically stable. The previous corollary then immediately gives the following result.

Corollary 1.3. *For varying $\pi \in \Pi_\phi(G)$, the Fargues-Scholze parameter φ_π depends only on ϕ.*

Of course, we expect that $\varphi_\pi = \phi$, but this seems to be a very difficult problem.

[1] As written, [FS24] in fact defines an analogous map $\mathfrak{Z}^{\mathrm{spec}}(G, \overline{\mathbf{Q}_\ell}) \to \mathfrak{Z}(G, \overline{\mathbf{Q}_\ell})$ for any fixed prime $\ell \neq p$ and any fixed algebraic closure $\overline{\mathbf{Q}_\ell}$, where the source is the ring of functions on the variety of semisimple L-parameters into ${}^L G(\overline{\mathbf{Q}_\ell})$, and the target is the center of the category of smooth $\overline{\mathbf{Q}_\ell}$-representations of $G(E)$. In this paper, we simply transport this map across a fixed choice of isomorphism $\iota : \overline{\mathbf{Q}_\ell} \xrightarrow{\sim} \mathbf{C}$, for some fixed $\ell \neq p$. However, by recent work of Scholze [Sch25], the resulting map is completely canonical and independent of the choices involved.

More generally, our main result immediately shows that for any group G for which the *existence of tempered L-packets for G* is known in the precise sense of [Var24, Hypothesis 2.5.1], the Fargues-Scholze parameter is constant on any such packet. By the results of [Art13, Mok15], this condition is satisfied for all quasisplit classical groups.

We can also say something about how the image of the map Ψ_G changes as G varies across inner forms. To explain this, note that $\mathfrak{Z}^{\mathrm{spec}}(G)$ depends only on the inner isomorphism class of G. In particular, if G^* is quasisplit and $b \in B(G^*)$ is a basic element with associated extended pure inner form $G := G^*_b$, Theorem 1.1 gives a pair of maps

$$\begin{array}{ccc} & & \mathfrak{Z}^{\mathrm{vst}}(G^*) \\ & \overset{\Psi_{G^*}}{\nearrow} & \\ \mathfrak{Z}^{\mathrm{spec}}(G^*) & & \\ & \underset{\Psi_G}{\searrow} & \\ & & \mathfrak{Z}^{\mathrm{vst}}(G) \end{array}$$

which of course factor over the relevant subrings $\mathfrak{Z}^{\mathrm{FS}}$. According to a conjecture of Scholze-Shin [SS13, Remark 6.4], we expect that Ψ_G is always surjective and that Ψ_{G^*} is an isomorphism. In particular, we expect there is a unique surjective ring map $\mathfrak{Z}^{\mathrm{vst}}(G^*) \to \mathfrak{Z}^{\mathrm{vst}}(G)$ compatible with the diagram above. The following theorem gives an unconditional substitute for this map.

Theorem 1.4. *If $\Psi_{G^*}(f) = 0$, then $\Psi_G(f) = 0$. In other words, there is a unique surjective $\mathfrak{Z}^{\mathrm{spec}}(G^*)$-algebra map $\tau_G : \mathfrak{Z}^{\mathrm{FS}}(G^*) \to \mathfrak{Z}^{\mathrm{FS}}(G)$. This map enjoys the following compatibilities.*

i. *If $M \subset G$ is any Levi subgroup, with corresponding Levi $M^* \subset G^*$, the diagram*

$$\begin{array}{ccc} \mathfrak{Z}^{\mathrm{FS}}(G^*) & \xrightarrow{\tau_G} & \mathfrak{Z}^{\mathrm{FS}}(G) \\ \downarrow & & \downarrow \\ \mathfrak{Z}^{\mathrm{FS}}(M^*) & \xrightarrow{\tau_M} & \mathfrak{Z}^{\mathrm{FS}}(M) \end{array}$$

commutes.

ii. *The map τ_G is compatible with the usual transfer map $\mathrm{Trans}_G : SD^{\mathrm{temp}}(G^*) \to SD^{\mathrm{temp}}(G)$ on stable tempered virtual characters, in the sense that $\mathrm{Trans}_G(z \cdot \Theta) = \tau_G(z) \cdot \mathrm{Trans}_G(\Theta)$ for all $z \in \mathfrak{Z}^{\mathrm{FS}}(G^*)$ and all $\Theta \in SD^{\mathrm{temp}}(G^*)$.*

More generally, if f^* and f are any compactly supported functions on $G^*(E)$ and $G(E)$ with matching stable orbital integrals, then $z * f^*$ and $\tau_G(z) * f$ have matching stable orbital integrals.

Here the vertical maps in i. are induced by the compatibility of Ψ_G with parabolic induction [FS24, Section IX.7.2], and we refer to the main text for reminders on the other terminologies here. Note that the existence of τ_G is not obviously related to stability, but our construction of this map crucially relies on Theorem 1.1. Part ii. is closely related to conjectures of Haines on "$\mathfrak{Z}$-transfer" for endoscopic groups [Hai14, Section 6.2], and essentially confirms his conjectures in the special case of extended pure inner forms. It is very likely that Theorem 1.4 could be easily extended to all inner forms by some simple argument with z-extensions, but we have not attempted this.

The proofs of these results are not very long, but they involve several different flavors of mathematics, so let us briefly highlight the key ingredients. One basic idea is that Hecke operators acting on sheaves on Bun_G give rise to certain extra endomorphisms $\mathscr{T}_\mu$ of the space of virtual characters $D(G)$ which *commute* with the action of elements of $\mathfrak{Z}^{\mathrm{FS}}(G)$ (see Lemma 2.4). This is a decategorification of the well-known principle that Hecke operators and excursion operators commute on $\mathcal{D}(\mathrm{Bun}_G, \overline{\mathbf{Q}_\ell})$. In principle, these endomorphisms could depend on our chosen isomorphism $\iota : \overline{\mathbf{Q}_\ell} \xrightarrow{\sim} \mathbf{C}$, but this dependence is actually harmless for our purposes.

We also adapt a wonderful idea from a recent paper of Chenji Fu [Fu24], who showed that Hecke eigensheaves on Bun_G at supercuspidal Fargues-Scholze parameters automatically give stable virtual characters at their stalks. The key observation here (recalled in a quantitative form in Lemma 2.6) is that the character formulas from [HKW22] show that as $\mu \to \infty$ in the appropriate sense, the limiting value of $\mathscr{T}_\mu$ on the regular elliptic set is a naive stable averaging.

Unfortunately, the character formulas in [HKW22] only give control over $\mathscr{T}_\mu$ on the regular elliptic set. The final key idea is to combine this control with Arthur's theory of elliptic tempered virtual characters [Art93, Art96]. The essential property of these gadgets is that they exactly span the subspace of tempered virtual characters which are fully controlled by their values on the regular elliptic set, and the complement of this subspace is spanned by parabolic inductions. Since the Fargues-Scholze map Ψ_G is compatible with parabolic induction, all together this gives precisely the right leverage to run arguments by induction on Levi subgroups. Although this aspect of Arthur's theory is certainly well-known in harmonic analysis, its use here in combination with the Fargues-Scholze machinery is new and seems to be very powerful. We will give some more applications of this technique elsewhere.

1.2 A word on terminology

In [Hai14] and [SS13], the term "stable Bernstein center" is used for the ring $\mathfrak{Z}^{\mathrm{spec}}(G)$. We prefer our current notation and (following [FS24]) the appellation "spectral Bernstein center", since the link between this ring and stability conditions is not clear a priori, and this also aligns very well with the general usage of the terms "spectral" and "geometric" for the two sides of the categorical local Langlands correspondence. The space $\mathfrak{Z}^{\mathrm{st}}(G)$ is denoted $\mathcal{Z}_1(G)$ in [Var24]; since the meaning of the word "stable" is fixed, there seems to be very little choice in name here, and our convention on this point agrees with [BKV15]. The ring $\mathfrak{Z}^{\mathrm{vst}}(G)$ is denoted $\mathcal{Z}_2(G)$ in [Var24] and $\mathcal{Y}^{\mathrm{st}}(G)$ in [SS13, Remark 6.4], and we hope our slightly uncreative use of the term "very stable" is acceptable.

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Finally, I learned how to think about modern harmonic analysis on p-adic groups by reading Sandeep Varma's paper [Var24]. It's a particular pleasure to acknowledge the influence of this paper, and to thank Sandeep for answering my many questions. I also sincerely thank the referee for their very detailed comments and suggestions which significantly improved this paper.

[2] More precisely, the crucial idea emerged in Frankfurt airport while waiting to board my flight home.

2 Proofs

2.1 Harmonic analysis

In this section we collect some standard results in harmonic analysis. We learned essentially all of this material from [Var24]. Our notational conventions on Haar measures, Levis, Weyl groups, spaces of characters, etc. mostly follow Arthur's paper [Art96] and also coincide with the conventions in [Var24]. In particular, we fix a minimal E-rational Levi $M_0 \subset G$, with absolute Weyl group $W_0 = N_G(M_0)(E)/M_0(E)$, and then write $\mathcal{L}$ the set of E-rational Levi subgroups $M \subset G$ which contain M_0 equipped with its natural W_0-action, $W(M) = N_G(M)(E)/M(E)$ for any $M \in \mathcal{L}$, etc. We also fix once and for all a Haar measure dm on $M(E)$ for all $M \in \mathcal{L}$. We write $G(E)_{\mathrm{ell}}$ for the set of strongly regular elliptic elements. All representations and function spaces are defined over the complex numbers.

Inside $C_c(G(E))$, we have the subspace $C_c(G(E))^{\mathrm{null}}$ of null functions f characterized by four equivalent conditions (see [Kaz86, Theorem 0], and see also [Dat00, Section 3.1] for a nice discussion):

- $\mathrm{tr}(f|\pi) = 0$ for all irreducible representations π.
- $\mathrm{tr}(f|\pi) = 0$ for all tempered irreducible representations π.
- All regular semisimple orbital integrals of f are zero.
- f is in the subspace of commutators, i.e. the linear span of functions of the form $h(x) - h(gxg^{-1})$.

The equivalence of these conditions is sometimes called the Kazhdan density theorem. We write $\mathcal{I}(G) = C_c(G(E))/C_c(G(E))^{\mathrm{null}}$. Note that the action of $\mathfrak{Z}(G)$ on $C_c(G(E))$ preserves null functions: if $f \in C_c(G(E))^{\mathrm{null}}$ and $z \in \mathfrak{Z}(G)$, then $\mathrm{tr}(z * f|\pi) = z_\pi \mathrm{tr}(f|\pi) = 0$ for all irreducible π (see e.g. [Hai14, Corollary 3.2.1]), so $z * f$ is null. Therefore the action descends to an action of $\mathfrak{Z}(G)$ on $\mathcal{I}(G)$ which we also denote as $z * f$.

For any Levi M, there is a canonical map $\mathfrak{Z}(G) \to \mathfrak{Z}(M)$, denoted $z \mapsto r_M(z)$ or just $z \mapsto r(z)$ if M is clear from context, defined as in [BDK86, Section 2.4]. There is also a canonical constant term map

$$\begin{aligned} &\mathcal{I}(G) \to \mathcal{I}(M) \\ &\quad f \mapsto f_M = \delta_P^{1/2}(m) \int_{U(E)} \int_K f(kmuk^{-1}) dk du \end{aligned}$$

which strictly speaking is defined on $C_c(G(E))$, but it descends to the quotient $\mathcal{I}(G)$. Here $P = MU$ is any parabolic with Levi M, $K \subset G(E)$ is any open compact with dk the normalized Haar measure, and du is determined by our choices of Haar measures on $G(E)$ and $M(E)$. This map is characterized by the formula $\mathrm{tr}(f_M|\pi) = \mathrm{tr}(f|i_M^G \pi)$ for irreducible π. It is easy to see from this formula that $(z * f)_M = r_M(z) * f_M$.

We write $\mathcal{I}(G)^{\mathrm{cusp}}$ for the subspace of *cuspidal* functions f characterized by the vanishing of f_M for all proper Levis M, or equivalently by the vanishing of all orbital integrals at non-elliptic regular semisimple elements. There is then a canonical decomposition

$$\mathcal{I}(G) = \bigoplus_{M \in \mathcal{L}/W_0} (\mathcal{I}(M)^{\mathrm{cusp}})^{W(M)} \tag{1}$$

due to Arthur [Art96, Section 4] (we learned this result from [Var24, Remark 4.2.7.(i)]).

Dually, let $\mathrm{Dist}(G)$ be the linear dual of $\mathcal{I}(G)$, so this is the space of all invariant distributions on G. Let $D(G) \subset \mathrm{Dist}(G)$ be the subspace of virtual characters, and let $D^{\mathrm{temp}}(G) \subset D(G)$ be the span of characters of tempered irreducible representations. The Bernstein center acts on $\mathrm{Dist}(G)$ and preserves $D(G)$ and $D^{\mathrm{temp}}(G)$. We write this action as $z \cdot \Theta$. This action is defined in terms of the canonical $\mathfrak{Z}(G)$-action on $\mathcal{I}(G)$ by the tautological formula $(z \cdot \Theta)(f) = \Theta(z * f)$. Of course, if $\Theta = \Theta_\pi$ is the character of an irreducible representation, then $z \cdot \Theta_\pi = z_\pi \Theta_\pi$.

Inside $D^{\mathrm{temp}}(G)$, we have the still smaller subspace $D^{\mathrm{ell}}(G)$ defined as the linear span of Arthur's elliptic tempered virtual characters $\Theta(\tau)$, $\tau \in T_{\mathrm{ell}}(G)$ (see [Art93, p. 76] for the notation). Then there is a canonical decomposition

$$D^{\mathrm{temp}}(G) = \bigoplus_{M \in \mathcal{L}/W_0} (D^{\mathrm{ell}}(M))^{W(M)} \tag{2}$$

where the inclusion of the M-indexed summand on the right-hand side is induced by the parabolic induction map $i_M^G : D(M) \to D(G)$ (see [MgW18, Proposition 2.12] for a proof). In particular, any $\Theta \in D^{\mathrm{temp}}(G)$ admits a unique decomposition $\Theta = \Theta^{\mathrm{ell}} + \Theta^{\mathrm{ind}}$ where $\Theta^{\mathrm{ell}} \in D^{\mathrm{ell}}(G)$ and Θ^{ind} is in the span of the M-indexed summands of (2) for proper Levis M. We will freely and crucially use the fact that the pointwise evaluation map

$$\begin{aligned} D^{\mathrm{ell}}(G) &\to C(G(E)_{\mathrm{ell}}) \\ \Theta &\mapsto \Theta|_{G(E)_{\mathrm{ell}}} \end{aligned}$$

is injective: for this, simply note that the kernel of the map

$$\begin{aligned} D^{\mathrm{temp}}(G) &\to C(G(E)_{\mathrm{ell}}) \\ \Theta &\mapsto \Theta|_{G(E)_{\mathrm{ell}}} \end{aligned}$$

consists exactly of elements Θ such that $\Theta = \Theta^{\mathrm{ind}}$ by [HKW22, Theorem C.1.1].

This decompositions (1) and (2) are perfectly dual to each other (see [Var24, Remark 4.2.7.(ii)-(iii)] for a nice discussion of this). In particular, any $f \in \mathcal{I}(G)$ admits a unique decomposition $f = f^{\mathrm{cusp}} + f^{\mathrm{nc}}$ such that f^{cusp} is cuspidal and $\Theta(f^{\mathrm{nc}}) = 0$ for all $\Theta \in D^{\mathrm{ell}}(G)$. Note that for any $\Theta \in D^{\mathrm{temp}}(G)$, $\Theta(f) = \Theta^{\mathrm{ell}}(f^{\mathrm{cusp}}) + \Theta^{\mathrm{ind}}(f^{\mathrm{nc}})$.

Lemma 2.1. *For any $f \in \mathcal{I}(G)$ and $z \in \mathfrak{Z}(G)$, $(z * f)^{\mathrm{cusp}} = z * f^{\mathrm{cusp}}$ and $(z * f)^{\mathrm{nc}} = z * f^{\mathrm{nc}}$.*

Proof. If f is cuspidal, then $(z*f)_M = r_M(z)*f_M = 0$ for all proper Levis M, so $z*f$ is cuspidal. On the other hand, the subset $D^{\mathrm{ell}}(G) \subset D(G)$ is stable under the $\mathfrak{Z}(G)$-action, because any $z \in \mathfrak{Z}(G)$ acts on any $\Theta(\tau), \tau \in T_{\mathrm{ell}}(G)$ through a scalar since all the irreducible characters occurring in a given $\Theta(\tau)$ are subquotients of a common parabolic induction. More precisely, writing $\tau = (M, \sigma, r)$ as in [Art93], $\Theta(\tau)$ lies the span of the irreducible constituents of $i_M^G \sigma$, and any $z \in \mathfrak{Z}(G)$ acts on these constituents through a common scalar, namely the scalar via which $r_M(z)$ acts on σ. Therefore $\Theta(z * f^{\mathrm{nc}}) = (z \cdot \Theta)(f^{\mathrm{nc}}) = 0$ for all $\Theta \in D^{\mathrm{ell}}(G)$, so $z * f^{\mathrm{nc}}$ has vanishing cuspidal part. □

All of the spaces of virtual characters defined above have stable analogues, denoted SD, SD^{temp}, SD^{ell}. More precisely, for $? \in \{\emptyset, \mathrm{temp}, \mathrm{ell}\}$, $SD^?(G) \subset D^?(G)$ is the subspace of virtual characters Θ such that $\Theta(g) = \Theta(g')$ for all strongly regular semisimple elements $g, g' \in G(E)$ which are conjugate in $G(\overline{E})$. The decomposition (2) of $D^{\mathrm{temp}}(G)$ above admits a compatible stable analogue

$$SD^{\mathrm{temp}}(G) = \bigoplus_{M \in \mathcal{L}/W_0} (SD^{\mathrm{ell}}(M))^{W(M)} \tag{3}$$

as in [Var24, Proposition 3.2.8]. The elliptic inner product determines a canonical projection $D^{\text{ell}}(G) \to SD^{\text{ell}}(G)$ splitting the obvious inclusion, as in [Var24, Lemma 3.4.5] and the discussion preceding it, and the direct sum of these projections over $M \in \mathcal{L}/W_0$ yields an analogous projection $D^{\text{temp}}(G) \to SD^{\text{temp}}(G)$.

Recall that a function $f \in \mathcal{I}(G)$ is *unstable* if all its stable orbital integrals vanish. It is enough to impose this vanishing at strongly regular semisimple elements. We will need the result of Arthur that an element $f \in \mathcal{I}(G)$ is unstable iff $\Theta(f) = 0$ for all $\Theta \in SD^{\text{temp}}(G)$. For quasisplit groups this is explicitly proved in [Art96], and for general groups it is [Var24, Proposition 3.2.10]. This can be reformulated as follows.

Lemma 2.2. *The following conditions on an element $z \in \mathfrak{Z}(G)$ are equivalent.*

i. *For all unstable f, $z * f$ is unstable.*
ii. *The endomorphism $z\cdot$ of $D(G)$ preserves $SD(G)$.*
iii. *The endomorphism $z\cdot$ of $D^{\text{temp}}(G)$ preserves $SD^{\text{temp}}(G)$.*

Proof. Clearly i. implies ii. implies iii. That iii. implies i. is exactly the result of Arthur quoted before the lemma. □

As in the introduction, we call elements of the Bernstein center satisfying these equivalent conditions *very stable*.

By Kazhdan's density theorem, it is easy to see that $D^{\text{temp}}(G)$ is a faithful $\mathfrak{Z}(G)$-module. Indeed, if $z \in \mathfrak{Z}(G)$ kills all $\Theta \in D^{\text{temp}}(G)$, then in particular $\Theta_\pi(z * f) = (z \cdot \Theta_\pi)(f) = 0$ for all f and all tempered irreducible π, so then $z * f$ is a null function by Kazhdan density, and thus $z(f) = (z * f)(1) = 0$ for all f, so $z = 0$. This result has a straightforward stable analogue.

Lemma 2.3. *Under the natural action, $SD^{\text{temp}}(G)$ is a faithful $\mathfrak{Z}^{\text{vst}}(G)$-module.*

Proof. Let $z \in \mathfrak{Z}^{\text{vst}}(G)$ be an element such that $z \cdot \Theta = 0$ for all stable tempered virtual characters Θ. Then $\Theta(z * f) = 0$ for all f and all such Θ, so by Arthur's result recalled above, $z * f$ has vanishing stable orbital integrals for all f. Now the invariant distribution $\delta : h \to h(1)$ is stable [Kot88, Proposition 1], i.e. it is in the closed linear span of stable orbital integrals, so $0 = \delta(z * f) = z(f)$ for all f. Therefore $z = 0$ as desired. □

2.2 Excursion versus Hecke

The key extra symmetry of elements $z \in \mathfrak{Z}^{\text{FS}}(G)$ which will enforce their stability is their commutation with certain endomorphisms of $D(G)$ coming from Hecke operators on Bun_G. To explain this, let $\mathcal{D}(\text{Bun}_G) = \mathcal{D}_{\text{lis}}(\text{Bun}_G, \overline{\mathbf{Q}_\ell})$ be the category of sheaves defined in [FS24, Section VII.7]. There is an open immersion $i_1 : [*/\underline{G(E)}] \to \text{Bun}_G$, which induces natural functors $i_{1!} : \mathcal{D}(G(E), \overline{\mathbf{Q}_\ell}) \to \mathcal{D}(\text{Bun}_G)$ and $i_1^* : \mathcal{D}(\text{Bun}_G) \to \mathcal{D}(G(E), \overline{\mathbf{Q}_\ell})$. Moreover, for any ireducible $\hat{G}$-representation V, there is a natural Hecke operator $T_V : \mathcal{D}(\text{Bun}_G) \to \mathcal{D}(\text{Bun}_G)$ defined as in [FS24, Section IX.2].

Now fix a conjugacy class of cocharacters $\mu : \mathbf{G}_{m,\overline{E}} \to G_{\overline{E}}$ such that $1 \in B(G, \mu)$, or equivalently such that $V_\mu|_{Z(\hat{G})^\Gamma}$ is trivial, where V_μ is the irreducible representation of $\hat{G}$ with highest weight μ. Then $i_1^* T_{V_\mu^*} i_{1!}$ defines an endofunctor of $\mathcal{D}(G(E), \overline{\mathbf{Q}_\ell})$, which by [HKW22, Proposition 6.4.5] preserves the finite length objects and hence induces an endomorphism on the Grothendieck group $K_0\mathrm{Rep}_{\mathrm{fl}}(G(E), \overline{\mathbf{Q}_\ell})$. Transporting this endomorphism across our fixed isomorphism ι and $\mathbf{C}$-linearizing, we get an endomorphism $\mathscr{T}_\mu : D(G) \to D(G)$.[3]

Lemma 2.4. *For any $z \in \mathfrak{Z}^{\mathrm{FS}}(G)$, $z\cdot$ and $\mathscr{T}_\mu$ commute as endomorphisms of $D(G)$.*

This commutation of $z\cdot$ and $\mathscr{T}_\mu$ is the crucial extra symmetry we will exploit.

Proof. Fix any irreducible representation π, so then $z \cdot \pi = z_\pi \pi$ for some $z_\pi \in \mathbf{C}$. Write $\mathscr{T}_\mu(\Theta_\pi) = \sum n_i \Theta_{\pi_i}$. Since excursion operators are built from Hecke operators, and Hecke operators commute with each other, excursion operators commute with Hecke operators, and in particular $z \cdot \pi_i = z_\pi \pi_i$ for all i. Therefore

$$\begin{aligned} z \cdot \mathscr{T}_\mu(\Theta_\pi) &= \sum n_i z \cdot \Theta_{\pi_i} \\ &= z_\pi \sum n_i \Theta_{\pi_i} \\ &= z_\pi \mathscr{T}_\mu(\Theta_\pi) \\ &= \mathscr{T}_\mu(z \cdot \Theta_\pi), \end{aligned}$$

so $\mathscr{T}_\mu(z \cdot -) - z \cdot \mathscr{T}_\mu(-)$ annihilates Θ_π for every irreducible π. Since the Θ_π's form a basis of $D(G)$, this gives the result. □

We will also need some very non-formal facts about the operator $\mathscr{T}_\mu$. These all follow from the main results of [HKW22], which give an explicit formula for the restriction of $\mathscr{T}_\mu(\Theta)$ to $G(E)_{\mathrm{ell}}$ for any Θ. We now recall this formula. Fix any $g \in G(E)_{\mathrm{ell}}$ with centralizer T_g, and let $[[g]]$ denote the set of conjugacy classes in the stable conjugacy class of g. For any element $g' \in [[g]]$, we defined ([HKW22, Definition 3.2.2]) a certain invariant $\mathrm{inv}(g, g') \in B(T_g) = X_*(T_g)_\Gamma$. This invariant has the property that for each $\lambda \in X_*(T_g)$ such that $\dim V_\mu[\lambda] \neq 0$, there is *exactly* one element $g' \in [[g]]$ such that $\overline{\lambda} = \mathrm{inv}(g, g')$ in $X_*(T_g)_\Gamma$, where $\overline{\lambda}$ is the natural projection of λ along $X_*(T_g) \to X_*(T_g)_\Gamma$. Indeed, the uniqueness of g' is [HKW22, Remark 3.2.5], and existence follows from [Fu24, Lemma 4.2.7].

In this notation, the character formula proved in [HKW22, Theorem 6.5.2] says that

$$\mathscr{T}_\mu(\Theta)(g) = \sum_{\substack{\lambda \in X_*(T_g),\, g' \in [[g]] \\ \mathrm{inv}(g,g') = \overline{\lambda}}} \dim V_\mu[\lambda] \cdot \Theta(g')$$

for any $\Theta \in D(G)$. Note that the sign $(-1)^d$ appearing in [HKW22, Definition 3.2.7] disappears in the present situation, since this sign agrees with $e(G)e(G_b)$ by [HKW22, Equation 3.3.3] and we are in a scenario where $G_b = G$.

We record a few consequences of this result.

Proposition 2.5. *Fix μ as above.*

i. *If $\Theta \in D(G)$ is parabolically induced, then also $\mathscr{T}_\mu(\Theta)$ is parabolically induced.*

ii. *If $\Theta \in D(G)$ is stable, then $\mathscr{T}_\mu(\Theta) = \dim V_\mu \cdot \Theta + \Theta'$ where Θ' is parabolically induced.*

[3]This endomorphism depends on our fixed isomorphism $\iota : \mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}_\ell}$, which we suppress from the notation. Note however that by the main results of [HKW22] recalled below, the restriction of $\mathscr{T}_\mu(\Theta)$ to $G(E)_{\mathrm{ell}}$ is completely independent of this choice.

Proof. In general, $\Theta \in D(G)$ is parabolically induced if and only if $\Theta|_{G(E)_{\mathrm{ell}}}$ vanishes identically (see [HKW22, Theorem C.1.1]). Part i. then follows from the formula for the character of $\mathscr{T}_\mu(\Theta)$ at elliptic elements recalled above, since if $\Theta|_{G(E)_{\mathrm{ell}}} = 0$ then visibly $\mathscr{T}_\mu(\Theta)|_{G(E)_{\mathrm{ell}}} = 0$. Part ii. follows similarly, since if Θ is stable then

$$\begin{aligned}\mathscr{T}_\mu(\Theta)(g) &= \sum_{\substack{\lambda \in X_*(T_g),\, g' \in [[g]] \\ \mathrm{inv}(g,g') = \overline{\lambda}}} \mathrm{dim} V_\mu[\lambda] \cdot \Theta(g') \\ &= \sum_{\lambda \in X_*(T_g)} \mathrm{dim} V_\mu[\lambda] \cdot \Theta(g) \\ &= \dim V_\mu \cdot \Theta(g)\end{aligned}$$

for all $g \in G(E)_{\mathrm{ell}}$, so $\mathscr{T}_\mu(\Theta) - \mathrm{dim} V_\mu \cdot \Theta$ vanishes identically on $G(E)_{\mathrm{ell}}$. □

We will also adapt a marvelous idea from Chenji Fu's paper [Fu24], showing that as $\mu \to \infty$ in an appropriate sense, $\mathscr{T}_\mu$ implements a *stable averaging*. We will need a version of this result with some uniformity in g. To explain this, set $\mu_m = 4m\rho_G$ for $m \geq 1$, where $2\rho_G$ is the usual sum of positive roots. Now for any conjugation-invariant function $\phi \in C(G(E)_{\mathrm{ell}})^{G(E)}$, define its stable average by the formula $\phi_{\mathrm{st}}(g) = \frac{1}{|[[g]]|} \sum_{g' \in [[g]]} \phi(g')$ for all $g \in G(E)_{\mathrm{ell}}$.

Lemma 2.6. *Fix any $\Theta \in D(G)$, and fix $U \subset G(E)$ a compact subset whose elliptic part is stably invariant in the weak sense that for all $g \in G(E)_{\mathrm{ell}}$, either $[[g]] \cap U = 0$ or U meets every conjugacy class in $[[g]]$. Then*

$$|D(g)|^{1/2} |\frac{1}{\mathrm{dim} V_{\mu_m}} \mathscr{T}_{\mu_m}(\Theta)(g) - \Theta_{\mathrm{st}}(g)|$$

tends to zero uniformly *in $g \in G(E)_{\mathrm{ell}} \cap U$ as $m \to \infty$.*

Here $|D(g)|$ is the usual Weyl discriminant.

Proof. For any given $g \in G(E)_{\mathrm{ell}}$, we can rearrange the character formula as

$$\mathscr{T}_\mu(\Theta)(g) = \sum_{g' \in [[g]]} \Theta(g') \sum_{\substack{\lambda \in X_*(T_g) \\ \mathrm{inv}(g,g') = \overline{\lambda}}} \mathrm{dim} V_\mu[\lambda].$$

Defining the rational number

$$C_m(g, g') = \frac{1}{\mathrm{dim} V_{\mu_m}} \sum_{\substack{\lambda \in X_*(T_g) \\ \mathrm{inv}(g,g') = \overline{\lambda}}} \mathrm{dim} V_{\mu_m}[\lambda],$$

we can trivially rearrange this further to get

$$\frac{1}{\mathrm{dim} V_{\mu_m}} \mathscr{T}_{\mu_m}(\Theta)(g) = \sum_{g' \in [[g]]} \Theta(g') C_m(g, g').$$

On the other hand, Fu's analysis (specifically, the proof of [Fu24, Theorem 4.3.1]) shows that for sufficiently large m, we have $|C_m(g, g') - \frac{1}{|[[g]]|}| \leq \frac{C}{m}$ for some fixed constant C which depends only on the ambient group G. Then we get that $\frac{1}{\dim V_{\mu_m}}\mathscr{T}_{\mu_m}(\Theta)(g) \to \Theta_{\mathrm{st}}(g)$ pointwise on $G(E)_{\mathrm{ell}}$, and in fact that

$$|\frac{1}{\dim V_{\mu_m}}\mathscr{T}_{\mu_m}(\Theta)(g) - \Theta_{\mathrm{st}}(g)| \leq \frac{C}{m}\sup_{x\in[[g]]}|\Theta(x)|$$

for all $g \in G(E)_{\mathrm{ell}}$ where C depends only on G. Since the Weyl discriminant is invariant under stable conjugacy, we can insert it into the above estimate, giving

$$|D(g)|^{1/2}|\frac{1}{\dim V_{\mu_m}}\mathscr{T}_{\mu_m}(\Theta)(g) - \Theta_{\mathrm{st}}(g)| \leq \frac{C}{m}\sup_{x\in[[g]]}|D(x)|^{1/2}|\Theta(x)|.$$

Now by a deep theorem of Harish-Chandra, $|D(x)|^{1/2}|\Theta(x)|$ (extended by zero from $G(E)_{\mathrm{reg.ss}}$ to $G(E)$) is bounded on any compact subset of $G(E)$. (See e.g. [Clo87, Theorem 1] for a proof of a more general result.) Therefore, taking a further supremum over all $g \in G(E)_{\mathrm{ell}} \cap U$, the left side of the previous equation is bounded by $\frac{C}{m}\sup_{x\in U}|D(x)|^{1/2}|\Theta(x)| \leq \frac{C'}{m}$ as $m \to \infty$. This gives the claim. ∎

We will also need a slight generalization of some of these results, where the operator $\mathscr{T}_\mu$ now moves between inner forms. For this, pick *any* conjugacy class of cocharacters $\mu : \mathbf{G}_{m,\overline{E}} \to G_{\overline{E}}$, and let $b \in B(G,\mu)$ be the unique basic element, with G_b the associated extended pure inner form of G. Then by the same recipe as above, the functors $i_1^* T_{V_\mu^*} i_{b!}$ and $i_b^* T_{V_\mu} i_{1!}$ preserve finite length objects and induce linear maps $\mathscr{T}_\mu : D(G_b) \to D(G)$ and $\mathscr{T}_\mu^* : D(G) \to D(G_b)$.

Proposition 2.7. i. *If $f \in \mathfrak{Z}^{\mathrm{spec}}(G)$ is any element, with images $z = \Psi_G(f)$ and $z' = \Psi_{G_b}(f)$, then $\mathscr{T}_\mu(z' \cdot \Theta) = z \cdot \mathscr{T}_\mu(\Theta)$ for all $\Theta \in D(G_b)$, and $\mathscr{T}_\mu^*(z \cdot \Theta) = z' \cdot \mathscr{T}_\mu^*(\Theta)$ for all $\Theta \in D(G)$.*

ii. *If $G(E)_{\mathrm{ell}} \ni g \sim^{\mathrm{st}} g' \in G_b(E)_{\mathrm{ell}}$ is any stably conjugate pair of elliptic elements and $\Theta \in SD(G_b)$ is stable, then*

$$\mathscr{T}_\mu(\Theta)(g) = e(G)e(G_b)(\dim V_\mu)\Theta(g').$$

Likewise, if $\Theta \in SD(G)$ is stable, then

$$\mathscr{T}_\mu^*(\Theta)(g') = e(G)e(G_b)(\dim V_\mu)\Theta(g).$$

Proof. Part i. again follows from the commutation of Hecke operators with excursion operators. For part ii., we again need the character formula, but now in its most general form. Recall that for a fixed basic b, an elliptic element $g \in G(E)$, and an elliptic element $g' \in G_b(E)$ which is stably conjugate to g under the canonical identification $G(\overline{E}) = G_b(\overline{E})$, [HKW22, Definition 3.2.2] defines an invariant $\mathrm{inv}[b](g,g') \in B(T_g) = X_*(T_g)_\Gamma$. Let $[[g]]_b$ denote the set of conjugacy classes in $G_b(E)$ which are stably conjugate to g. In this notation, the character formula reads

$$\mathscr{T}_\mu(\Theta)(g) = e(G)e(G_b) \sum_{\substack{\lambda\in X_*(T_g),\, g'\in[[g]]_b \\ \mathrm{inv}[b](g,g')=\overline{\lambda}}} \dim V_\mu[\lambda] \cdot \Theta(g').$$

This again follows from [HKW22, Theorem 6.5.2], using [HKW22, Equation (3.3.3)] for the signs. Moreover, if λ is such that $\dim V_\mu[\lambda] \neq 0$, there is a unique g' such that $\mathrm{inv}[b](g,g') = \overline{\lambda}$, which again follows from [HKW22, Remark 3.2.5] and [Fu24, Lemma 4.2.7]. Thus if Θ is stable, the sum appearing in the character formula collapses to $(\dim V_\mu)\Theta(g')$, as claimed. This gives the first claim in part ii., and the second claim follows by an identical argument with the roles of G and G_b swapped. ∎

2.3 Stability

In this section we prove Theorem 1.1.

Let $z \in \mathfrak{Z}^{\mathrm{FS}}(G)$ be in the image of the Fargues-Scholze map Ψ_G. We need to prove that for any unstable $f \in \mathcal{I}(G)$, $z * f$ is unstable. For this, it is enough to see that $\Theta(z * f) = 0$ for all $\Theta \in SD^{\mathrm{temp}}(G)$ as recalled in Lemma 2.2 and the discussion preceding it. We will prove this by induction on the semisimple rank of G.

First suppose Θ is parabolically induced. By the decomposition (3) recalled in Section 2.1, without loss of generality we can assume $\Theta = i_M^G \Theta_M$ for some $\Theta_M \in SD^{\mathrm{temp}}(M)$ and some proper Levi M. Then

$$\begin{aligned}\Theta(z * f) &= (z \cdot i_M^G \Theta_M)(f). \\ &= i_M^G(r_M(z) \cdot \Theta_M)(f)\end{aligned}$$

where in the second line we used [BDK86, Proposition 2.4]. Now $r_M(z)$ is in the image of Ψ_M by compatibility of the Fargues-Scholze map with parabolic induction, so by induction on the semisimple rank and Lemma 2.2 we know that $r_M(z) \cdot \Theta_M$ is stable. Then also $i_M^G(r_M(z) \cdot \Theta_M)$ is stable, so its evaluation on the unstable function f vanishes.

This reduces us to the case where $\Theta \in SD^{\mathrm{ell}}(G)$. By Lemma 2.1, we can assume our unstable function f is cuspidal, in which case also $z * f$ is cuspidal. Now, with μ as in Lemma 2.4, consider the quantity

$$C_\mu := \frac{1}{\dim V_\mu} \mathscr{T}_\mu(\Theta)(z * f).$$

By Proposition 2.5, $\mathscr{T}_\mu(\Theta) = \dim V_\mu \cdot \Theta + \Theta'$ for some parabolically induced Θ'. Since $z * f$ is cuspidal, $\Theta'(z * f) = 0$, so this simplifies to $C_\mu = \Theta(z * f)$ which is evidently a constant independent of μ. Our goal is to show that this constant vanishes. Writing $\Xi = z \cdot \Theta$, Lemma 2.4 shows that

$$\begin{aligned}C_\mu &= \frac{1}{\dim V_\mu}(z \cdot \mathscr{T}_\mu(\Theta))(f) \\ &= \frac{1}{\dim V_\mu} \mathscr{T}_\mu(z \cdot \Theta)(f) \\ &= \frac{1}{\dim V_\mu} \mathscr{T}_\mu(\Xi)(f)\end{aligned}$$

for any μ. Note that although Θ is stable, Ξ certainly need not be stable a priori.[4]

At this point we use Fu's method. More precisely, taking $\mu = \mu_m$ with $m \to \infty$ as in Section 2.2, we will use that the operator $\frac{1}{\dim V_\mu} \mathscr{T}_\mu(\Xi)$ effects a stable averaging as discussed there. To implement this, for any $\Theta \in D^{\mathrm{temp}}(G)$, let $\Theta^{\mathrm{st}} \in SD^{\mathrm{temp}}(G)$ be its stable projection. Writing $\frac{1}{\dim V_{\mu_m}} \mathscr{T}_{\mu_m}(\Xi) = \Xi^{\mathrm{st}} + \Phi_{\mu_m}$, it is clear that $\Xi^{\mathrm{st}}(f) = 0$ since f is unstable, so $C_{\mu_m} = \Phi_{\mu_m}(f)$. We will now show that as $m \to \infty$, $\Phi_{\mu_m}(f) \to 0$.

To proceed further, we exploit the cuspidality of f to rewrite $\Phi_{\mu_m}(f)$ via a simple form of the Weyl integration formula. More precisely, fix a Haar measure da on the split center $A_G(E)$, and set $O_\gamma(f) = \int_{A_G(E)\backslash G(E)} f(x^{-1}\gamma x)dx$ as a function on $G(E)_{\mathrm{ell}}$, where $dx = dg/da$ in the usual manner. Then for any $\Theta \in D(G)$ and any $f \in \mathcal{I}(G)^{\mathrm{cusp}}$, the Weyl integration formula can be written as

$$\Theta(f) = \sum_T \frac{1}{|W(G,T)(E)|} \int_{T(E)} \Theta(t) O_t(f) |D(t)| dt.$$

[4] In fact, by Lemma 2.2, we are exactly trying to prove that $z\cdot$ preserves stability.

Here the sum runs over a (finite) set of representatives for the $G(E)$-conjugacy classes of elliptic maximal tori in G, and dt is the Haar measure on $T(E)$ determined by the chosen Haar measure on $A_G(E)$ and the normalized Haar measure on the compact group $T(E)/A_G(E)$. We briefly recall some facts about convergence. For each T, the set of elements $t \in T(E) \cap G(E)_{\text{ell}}$ such that $O_t(f) \neq 0$ has compact closure C_T in $T(E)$. Now, by fundamental results of Harish-Chandra, the function $|D(g)|^{1/2}\Theta(g)$ (extended by zero from the regular semisimple locus) is locally bounded on $G(E)$ (as recalled in Section 2.2), and $|D(\gamma)|^{1/2}O_\gamma(f)$ is a bounded function on $G(E)_{\text{ell}}$ (see e.g. [Art91, Section 4]). In particular, for a fixed cuspidal f and varying Θ, we can replace each integral above by an integral over the fixed compact subset $C_T \subset T(E)$, and the integrand is a bounded function on that compact subset and is locally constant on a dense open subset thereof.

Now substituting in Φ_{μ_m} for Θ in the Weyl integration formula above, we are reduced to showing that $|D(t)|^{1/2}\Phi_{\mu_m}(t) \to 0$ uniformly on C_T as $m \to \infty$. Here again, $|D(t)|^{1/2}\Phi_{\mu_m}(t)$ is defined a priori as a bounded function on $C_T \cap G(E)_{\text{ell}}$ and extended by zero to C_T. Recall that by definition, $\Phi_{\mu_m} = \frac{1}{\dim V_{\mu_m}}\mathscr{T}_{\mu_m}(\Xi) - \Xi^{\text{st}}$. First we compute the restriction of Ξ^{st} to $G(E)_{\text{ell}}$. This follows from some general theory: by [Var24, Lemma 3.4.5], $\Theta^{\text{st}}|_{G(E)_{\text{ell}}} = (\Theta|_{G(E)_{\text{ell}}})_{\text{st}}$ for any $\Theta \in D(G)$, where $f \mapsto f_{\text{st}}$ is the naive stable averaging discussed immediately before Lemma 2.6. In particular, this applies to Ξ, so we get

$$\Phi_{\mu_m}(g) = \frac{1}{\dim V_{\mu_m}}\mathscr{T}_{\mu_m}(\Xi)(g) - \Xi_{\text{st}}(g)$$

for any $g \in G(E)_{\text{ell}}$. Now choose a compact subset $U \subset G(E)$ whose elliptic part is stably invariant and which moreover contains C_T for each T. Then by Lemma 2.6, $|D(g)|^{1/2}\Phi_{\mu_m}(g) \to 0$ uniformly as $m \to \infty$ for all $g \in U \cap G(E)_{\text{ell}}$, and in particular for all $t \in C_T \cap G(E)_{\text{ell}}$. But then this immediately extends to the same statement for all $t \in C_T$ since $|D(g)|^{1/2}\Phi_{\mu_m}(g)$ is extended by zero from the regular semisimple locus.

Putting all of this together, we get that

$$\Phi_{\mu_m}(f) = \sum_T \frac{1}{|W(G,T)(E)|}\int_{C_T} |D(t)|^{1/2}\Phi_{\mu_m}(t) \cdot |D(t)|^{1/2}O_t(f)dt$$

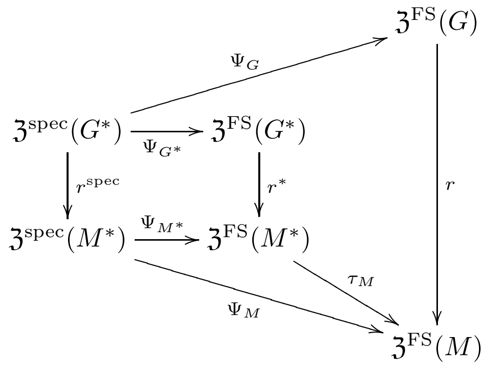
where $C_T \subset T(E)$ is compact, both halves of the integrand are bounded on C_T, and $|D(t)|^{1/2}\Phi_{\mu_m}(t) \to 0$ *uniformly* on C_T for each T as $m \to \infty$. Therefore $\Phi_{\mu_m}(f) \to 0$ as $m \to \infty$. This gives the result.

2.4 Inner forms

In this section we deal with Theorem 1.4. To construct the map τ_G, we proceed by induction on the semisimple rank. More precisely, fix some element $f \in \mathfrak{Z}^{\text{spec}}(G^*)$ such that $z := \Psi_G(f) \neq 0$. We need to show that $z^* := \Psi_{G^*}(f) \neq 0$.

By Lemma 2.3, $SD^{\text{temp}}(G)$ is a faithful $\mathfrak{Z}^{\text{vst}}(G)$-module, thus a faithful $\mathfrak{Z}^{\text{FS}}(G)$-module by Theorem 1.1. In particular, the endomorphism $z\cdot$ of $SD^{\text{temp}}(G)$ is not identically zero. Suppose first that $z \cdot \Theta \neq 0$ for some parabolically induced $\Theta = i_M^G\Theta_M$ with $\Theta_M \in SD^{\text{temp}}(M)$. Let $M^* \subset G^*$

be the Levi subgroup corresponding to M. We have a commutative diagram



where τ_M exists and is surjective by induction on the semisimple rank. Now by [BDK86, Proposition 2.4], the linear map $z \cdot i_M^G(-)$ coincides with the linear map $i_M^G(r(z) \cdot -)$. In particular, since $z \cdot i_M^G \Theta_M \neq 0$, we get that $r(z) \neq 0$. But then

$$\begin{aligned} r(z) &= \Psi_M(r^{\mathrm{spec}}(f)) \\ &= \tau_M(\Psi_{M^*}(r^{\mathrm{spec}}(f))) \\ &= \tau_M(r^*(\Psi_{G^*}(f))) \\ &= \tau_M(r^*(z^*)) \end{aligned}$$

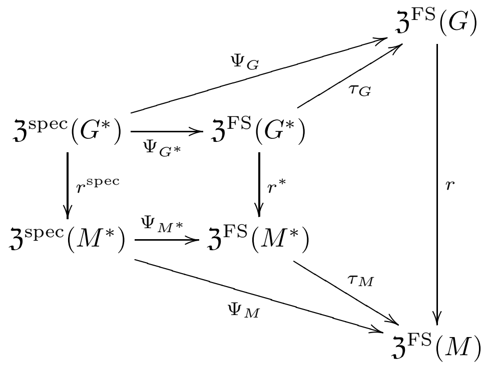
using the commutativity of the diagram, so $z^* \neq 0$.

It remains to deal with the case where z annihilates all parabolically induced elements of $SD^{\mathrm{temp}}(G)$. By Lemma 2.3, we may choose some $\Theta \in SD^{\mathrm{ell}}(G)$ such that $z \cdot \Theta \neq 0$. Pick some μ such that $b \in B(G^*, \mu)$, and let $\mathscr{T}_\mu : D(G) \to D(G^*)$ be the linear map induced by $i_1^* T_{V_\mu^*} i_{b!}$ (and our choice of isomorphism $\mathbf{C} \simeq \overline{\mathbf{Q}_\ell}$) as in the discussion preceding Proposition 2.7. By the commutation of Hecke operators with excursion operators, we get that $z^* \cdot \mathscr{T}_\mu(\Theta) = \mathscr{T}_\mu(z \cdot \Theta)$ as in Proposition 2.7.i. By assumption, $z \cdot \Theta \in SD^{\mathrm{ell}}(G)$ is nonzero, so it is not identically zero on $G(E)_{\mathrm{ell}}$. Now the character formula, Proposition 2.7.ii, shows that for all matching stably conjugate pairs $G^*(E)_{\mathrm{ell}} \ni g^* \sim^{\mathrm{st}} g \in G(E)_{\mathrm{ell}}$, we have an equality

$$\mathscr{T}_\mu(z \cdot \Theta)(g^*) = e(G) \dim V_\mu (z \cdot \Theta)(g),$$

so $\mathscr{T}_\mu(z \cdot \Theta)(g^*)$ is nonzero for some $g^* \in G^*(E)_{\mathrm{ell}}$. Therefore $\mathscr{T}_\mu(z \cdot \Theta) = z^* \cdot \mathscr{T}_\mu(\Theta) \neq 0$, so $z^* \neq 0$ as desired.

Next, for the compatibility with parabolic induction, note that we already have a diagram



where everything commutes except possibly the trapezoid spanned by r, r^*, τ_G, τ_M. But the surjectivity of Ψ_{G^*} immediately implies that this trapezoid commutes as well.

It remains to show compatibility with the transfer map. We first recall some properties of this map, referring to [Var24, Section 3.2] for details.[5] Fixing an inner twist and all other data as in [Var24, Section 3.2], we get a canonical injection $\mathcal{L}/W_0 \to \mathcal{L}^*/W_0^*$. The transfer map $\mathrm{Trans}_G : SD^{\mathrm{temp}}(G^*) \to SD^{\mathrm{temp}}(G)$ is then compatible with the grading

$$SD^{\mathrm{temp}}(G^*) = \bigoplus_{M^* \in \mathcal{L}^*/W_0^*} (SD^{\mathrm{ell}}(M^*))^{W(M^*)}$$

and its analogue for G in the following very strong sense.

- Its restriction to the summand $SD^{\mathrm{ell}}(G^*)$ factors over an isomorphism $\mathrm{Trans}_G^{\mathrm{ell}} : SD^{\mathrm{ell}}(G^*) \xrightarrow{\sim} SD^{\mathrm{ell}}(G)$ characterized by the equality $\mathrm{Trans}_G^{\mathrm{ell}}(\Theta)(g) = e(G)\Theta(g^*)$ for all matching stably conjugate pairs $G^*(E)_{\mathrm{ell}} \ni g^* \sim^{\mathrm{st}} g \in G(E)_{\mathrm{ell}}$.
- If $M^* \in \mathcal{L}^*/W_0^*$ is irrelevant in the sense that it is not the image of some $M \in \mathcal{L}/W_0$, Trans_G is identically zero on the M^*-indexed summand.
- If M^* is the image of some M, then $\mathrm{Trans}_G i_{M^*}^{G^*} \Theta_{M^*} = i_M^G \mathrm{Trans}_M^{\mathrm{ell}} \Theta_{M^*}$ for all $\Theta_{M^*} \in (SD^{\mathrm{ell}}(M^*))^{W(M^*)}$, compatibly with the Weyl equivariance via the appropriate identification $W(M) = W(M^*)$.

Now let $z \in \mathfrak{Z}^{\mathrm{FS}}(G^*)$ be any element, and pick any $\Theta \in SD^{\mathrm{temp}}(G^*)$. We need to show that $\mathrm{Trans}_G(z \cdot \Theta) = \tau_G(z) \cdot \mathrm{Trans}_G(\Theta)$. First suppose Θ is parabolically induced, say of the form $i_{M^*}^{G^*}\Xi$ for some $\Xi \in SD^{\mathrm{ell}}(M^*)$. Then by the remarks on the grading above and compatibility of the Bernstein center action with parabolic induction, we compute that

$$\mathrm{Trans}_G(z \cdot i_{M^*}^{G^*}\Xi) = \mathrm{Trans}_G(i_{M^*}^{G^*}(r^*(z) \cdot \Xi)).$$

If M^* is irrelevant, this is identically zero, as is $\mathrm{Trans}_G(\Theta)$, so there is nothing to prove. If M^* is

[5] Our convention differs from Varma's in one place only: we normalize the transfer factor between G and G^* to be the Kottwitz sign $e(G)$, rather than the scalar 1 as in [Var24].

relevant, we compute further that

$$\begin{aligned}\mathrm{Trans}_G(i_{M^*}^{G^*}(r^*(z)\cdot\Xi)) &= i_M^G\mathrm{Trans}_M^{\mathrm{ell}}(r^*(z)\cdot\Xi)\\ &= i_M^G\tau_M(r^*(z))\cdot\mathrm{Trans}_M^{\mathrm{ell}}\Xi\\ &= i_M^G r(\tau_G(z))\cdot\mathrm{Trans}_M^{\mathrm{ell}}\Xi\\ &= \tau_G(z)\cdot i_M^G\mathrm{Trans}_M^{\mathrm{ell}}\Xi\\ &= \tau_G(z)\cdot\mathrm{Trans}_G(\Theta),\end{aligned}$$

where the second equality follows by induction on the semisimple rank.

This reduces us to the case that $\Theta\in SD^{\mathrm{ell}}(G^*)$. Here we just need to see that $(\tau(z)\cdot\mathrm{Trans}_G^{\mathrm{ell}}(\Theta))(g)=\mathrm{Trans}_G^{\mathrm{ell}}(z\cdot\Theta)(g)$ for all $g\in G(E)_{\mathrm{ell}}$. Now we exploit [HKW22] one more time. More precisely, defining $\mathscr{T}_\mu^*: D(G^*)\to D(G)$ as the map induced by $i_b^*T_{V_\mu}i_{i!}$ (with b and μ as above), the character formula as in Proposition 2.7.ii shows that

$$\mathscr{T}_\mu^*(\Theta)(g)=\dim V_\mu\cdot\mathrm{Trans}_G^{\mathrm{ell}}(\Theta)(g)$$

for all $g\in G(E)_{\mathrm{ell}}$. But again, Hecke operators commute with excursion operators, so we know that $\mathscr{T}_\mu^*(z\cdot\Theta)=\tau_G(z)\cdot\mathscr{T}_\mu^*(\Theta)$. Evaluating both sides of this equality on any $g\in G(E)_{\mathrm{ell}}$ and invoking the character formula, we get the result.

Finally, suppose f^* and f are any matching functions. Fix any $z\in\mathfrak{Z}^{\mathrm{FS}}(G^*)$, and let $(\tau_G(z)*f)^*$ be a function on $G^*(E)$ matching $\tau_G(z)*f$. We need to see that $h:=z*f^*-(\tau_G(z)*f)^*$ is unstable. For this, pick any $\Theta\in SD^{\mathrm{temp}}(G^*)$. We then compute

$$\begin{aligned}\Theta(z*f^*) &= (z\cdot\Theta)(f^*)\\ &= \mathrm{Trans}_G(z\cdot\Theta)(f)\\ &= (\tau_G(z)\cdot\mathrm{Trans}_G\Theta)(f)\\ &= \mathrm{Trans}_G(\Theta)(\tau_G(z)*f)\\ &= \Theta\left((\tau_G(z)*f)^*\right)\end{aligned}$$

where the first and fourth equalities are trivial, the second and fifth equalities follow from the definition of the transfer map, and the third equality follows from our results so far. Therefore $\Theta(h)=0$, and since $\Theta\in SD^{\mathrm{temp}}(G^*)$ is arbitrary, this implies that h is unstable, as desired.

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